What is the distance to the CMB?

Ruth Durrer
Université de Genève
Départment de Physique Théorique and Center for Astroparticle Physics

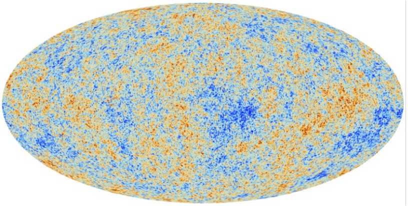
Work in collaboration with:
Chris Clarkson, Obinna Umeh and Roy Maartens, arXiv:1405.7860v2

Genève, septembre 19, 2014

Outline

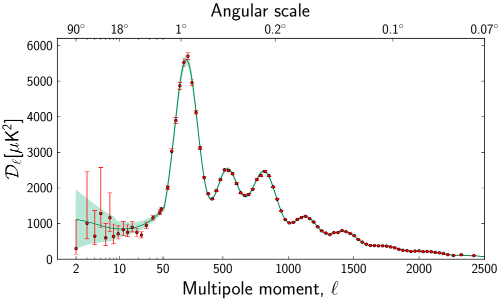
Introduction: The CMB

CMB sky as seen by Planck



$$\left\langle \frac{\Delta T}{T}(\mathbf{n})\frac{\Delta T}{T}(\mathbf{n}')\right\rangle = \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell P_\ell(\mathbf{n}\cdot\mathbf{n}')$$

$$D_\ell = \ell(\ell+1)C_\ell/(2\pi)$$



The Planck Collaboration: Planck results 2013 XV

Introduction

The CMB is our most precise cosmological data set. We know the angle subtended by the first peak to exquisite precision, (Planck Collaboration: Planck results 2013 XVI)

$$\theta_s = \frac{r_s}{d_A(z_s)} = (1.04131 \pm 0.00062) \times 10^{-2} \,.$$

Introduction

The CMB is our most precise cosmological data set. We know the angle subtended by the first peak to exquisite precision, (Planck Collaboration: Planck results 2013 XVI)

$$\theta_s = \frac{r_s}{d_A(z_s)} = (1.04131 \pm 0.00062) \times 10^{-2} \,.$$

The sound horizon at last scattering is (Planck Collaboration: Planck results 2013 XVI)

$$r_s = (144.71 \pm 0.60)\mathrm{Mpc} \,.$$

Introduction

The CMB is our most precise cosmological data set. We know the angle subtended by the first peak to exquisite precision, (Planck Collaboration: Planck results 2013 XVI)

$$\theta_s = \frac{r_s}{d_A(z_s)} = (1.04131 \pm 0.00062) \times 10^{-2}\,.$$

The sound horizon at last scattering is (Planck Collaboration: Planck results 2013 XVI)

$$r_s = (144.71 \pm 0.60)\text{Mpc}\,.$$

This relative error of 4.1×10^{-3} is also the precision with which we know $d_A(z_s)$.

Introduction

The CMB is our most precise cosmological data set. We know the angle subtended by the first peak to exquisite precision, (Planck Collaboration: Planck results 2013 XVI)

$$\theta_s = \frac{r_s}{d_A(z_s)} = (1.04131 \pm 0.00062) \times 10^{-2} \,.$$

The sound horizon at last scattering is (Planck Collaboration: Planck results 2013 XVI)

$$r_s = (144.71 \pm 0.60)\mathrm{Mpc} \,.$$

This relative error of 4.1×10^{-3} is also the precision with which we know $d_A(z_s)$.

Why are these quantities so well determined by the CMB?

Fluctuations of the CMB and cosmological parameters: acoustic peaks

Initial fluctuations from inflation are supposed to be generated with a very simple power spectrum,

$$k^3\langle\Psi(\mathbf{k})\Psi(\mathbf{k}')\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')\Delta_{\mathcal{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1} .$$

Here Ψ is the Bardeen potential.

Fluctuations of the CMB and cosmological parameters: acoustic peaks

Initial fluctuations from inflation are supposed to be generated with a very simple power spectrum,

$$k^3\langle\Psi(\mathbf{k})\Psi(\mathbf{k}')\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')\Delta_{\mathcal{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1} .$$

Here Ψ is the Bardeen potential.

The evolution of the Bardeen potential during the radiation dominated era is just a damped wave equation,

$$\ddot\Psi + 3\mathcal{H}(1+c_s^2)\dot\Psi + [3(c_s^2-w)\mathcal{H}^2 + c_s^2k^2]\Psi = 0\,.$$

The (growing mode) solution is constant on super Hubble scales, $k<\mathcal{H}$, and decays (like $\mathcal{H}^2$) and oscillates inside the horizon with wave number c_sk.

Fluctuations of the CMB and cosmological parameters: acoustic peaks

Initial fluctuations from inflation are supposed to be generated with a very simple power spectrum,

$$k^3\langle\Psi(\mathbf{k})\Psi(\mathbf{k}')\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')\Delta_{\mathcal{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1} .$$

Here Ψ is the Bardeen potential.

The evolution of the Bardeen potential during the radiation dominated era is just a damped wave equation,

$$\ddot\Psi + 3\mathcal{H}(1+c_s^2)\dot\Psi + [3(c_s^2-w)\mathcal{H}^2 + c_s^2k^2]\Psi = 0\,.$$

The (growing mode) solution is constant on super Hubble scales, $k < \mathcal{H}$, and decays (like $\mathcal{H}^2$) and oscillates inside the horizon with wave number c_sk.

The radiation density fluctuation is simply $\delta_r = \frac{2}{3}(k/\mathcal{H})^2\Psi$ and $\Delta T/T = \delta_r/4$.
The density fluctuations grow on super Hubble scales and oscillate at constant amplitude inside the horizon.

Fluctuations of the CMB and cosmological parameters: acoustic peaks

Initial fluctuations from inflation are supposed to be generated with a very simple power spectrum,

$$k^3\langle\Psi(\mathbf{k})\Psi(\mathbf{k}')\rangle = 2\pi^2\delta(\mathbf{k}-\mathbf{k}')\Delta_{\mathcal{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1} .$$

Here Ψ is the Bardeen potential.

The evolution of the Bardeen potential during the radiation dominated era is just a damped wave equation,

$$\ddot\Psi + 3\mathcal{H}(1+c_s^2)\dot\Psi + [3(c_s^2-w)\mathcal{H}^2 + c_s^2k^2]\Psi = 0 .$$

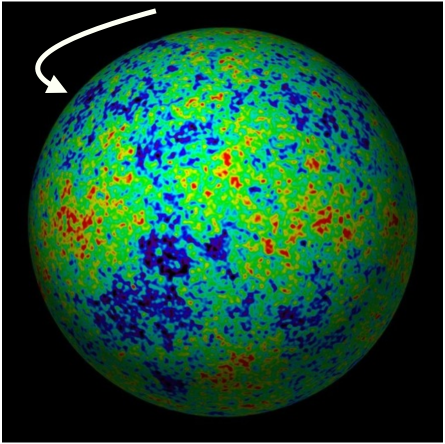
The (growing mode) solution is constant on super Hubble scales, $k < \mathcal{H}$, and decays (like $\mathcal{H}^2$) and oscillates inside the horizon with wave number c_sk.

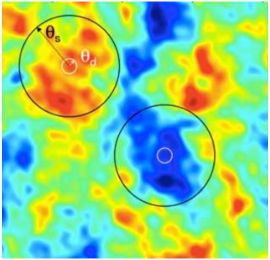
The radiation density fluctuation is simply $\delta_r = \frac{2}{3}(k/\mathcal{H})^2\Psi$ and $\Delta T/T = \delta_r/4$.
The density fluctuations grow on super Hubble scales and oscillate at constant amplitude inside the horizon.

The CMB is a snapshot of these oscillations at the moment of decoupling.

Acoustic peaks in the CMB

proper area $A \equiv 4\pi D^2_{\rm CMB} = \int_{4\pi} d_A^2(z,\theta)d\Omega$

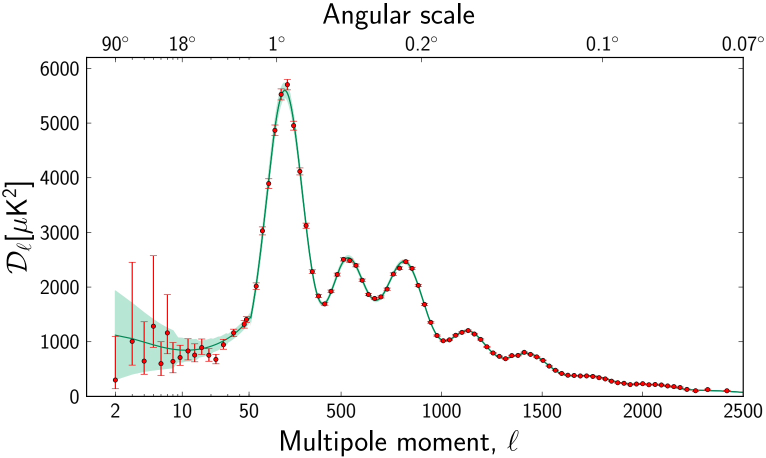




sound horizon scale
determined by

$$r_s = \theta_s D_{CMB} > \theta_s \bar{D}$$

Acoustic peaks in the CMB



In the flat sky approximation $\ell = kd_A$.
The spectrum strongly depends on: n_s, $\Delta_{\mathcal{R}}$, $\omega_b = \Omega_b h^2$, $\omega_m = \Omega_m h^2$, d_A.

Fluctuations of the CMB and cosmological parameters: the sound horizon and the distance to the CMB

The position of the first acoustic peak is given by $\theta_s = r_s/d_A(z_s)$ where

$$\frac{H_0}{h}(1+z_s)r_s = \frac{2}{\sqrt{3r\omega_m}}\log\left(\frac{\sqrt{1+z_s+r}+\sqrt{\frac{(1+z_s)r\omega_r}{\omega_m}+r}}{\sqrt{1+z_s}\left(1+\sqrt{\frac{r\omega_r}{\omega_m}}\right)}\right)\,, \qquad r = \frac{3\omega_b}{4\omega_\gamma}\,.$$

This expression is independent of h ($h/H_0 = (100\text{km/s/Mpc})^{-1}2997.9\text{Mpc}$)

Fluctuations of the CMB and cosmological parameters: the sound horizon and the distance to the CMB

The position of the first acoustic peak is given by $\theta_s = r_s/d_A(z_s)$ where

$$\frac{H_0}{h}(1+z_s)r_s = \frac{2}{\sqrt{3r\omega_m}}\log\left(\frac{\sqrt{1+z_s+r}+\sqrt{\frac{(1+z_s)r\omega_r}{\omega_m}+r}}{\sqrt{1+z_s}\left(1+\sqrt{\frac{r\omega_r}{\omega_m}}\right)}\right), \qquad r = \frac{3\omega_b}{4\omega_\gamma}.$$

This expression is independent of h ($h/H_0 = (100\text{km/s/Mpc})^{-1}2997.9\text{Mpc}$) while ($K=0$)

$$\frac{H_0}{h}(1+z_s)d_A(z_s) = \int_0^{z_s} dz\frac{H_0}{hH(z)} = \int_0^{z_s}\frac{dz}{[\omega_m(1+z)^3-\omega_m+h^2]^{1/2}}.$$

(for a flat ΛCDM universe) depends strongly on h,

Fluctuations of the CMB and cosmological parameters: the sound horizon and the distance to the CMB

The position of the first acoustic peak is given by $\theta_s = r_s/d_A(z_s)$ where

$$\frac{H_0}{h}(1+z_s)r_s = \frac{2}{\sqrt{3r\omega_m}}\log\left(\frac{\sqrt{1+z_s+r}+\sqrt{\frac{(1+z_s)r\omega_r}{\omega_m}+r}}{\sqrt{1+z_s}\left(1+\sqrt{\frac{r\omega_r}{\omega_m}}\right)}\right)\,, \qquad r = \frac{3\omega_b}{4\omega_\gamma}\,.$$

This expression is independent of h ($h/H_0 = (100\text{km/s/Mpc})^{-1}2997.9\text{Mpc}$) while ($K=0$)

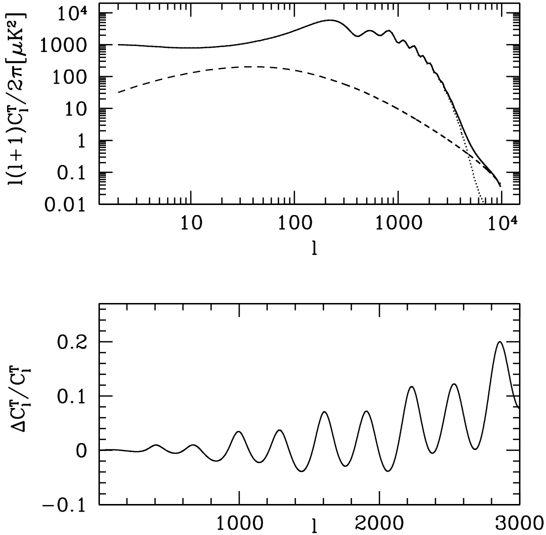
$$\frac{H_0}{h}(1+z_s)d_A(z_s) = \int_0^{z_s} dz\,\frac{H_0}{hH(z)} = \int_0^{z_s}\frac{dz}{[\omega_m(1+z)^3-\omega_m+h^2]^{1/2}}\,.$$

(for a flat ΛCDM universe) depends strongly on h,

$$\frac{\partial d_A(z_s)}{\partial h} \simeq -5\frac{d_A(z_s)}{h}$$

for standard Planck values. Hence a 1% error in d_A leads to a 5% error in h!

Lensing of the CMB



(Figure from RD, 'The Cosmic Microwave Background', 2008)

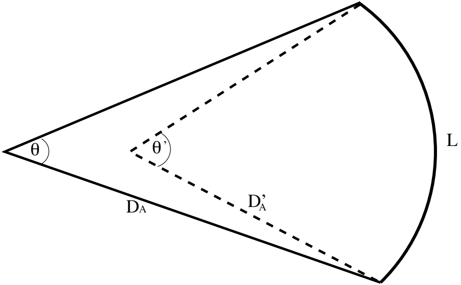
It is well known that lensing by foreground structure affects the CMB. This is taken into account by including lensing deflection,

$$\delta \mathbf{n} = \nabla \psi$$

i.e. 1st order lensing.

Fluctuations of the CMB and cosmological parameters: the distance to the CMB II

The acoustic oscillations of density fluctuations in the relativistic plasma begin in a minimum at the end of inflation. At the moment of decoupling they are $\propto -\cos(c_s k t_{\rm dec})$. The fluctuations with wavelength $2r_s = 2\pi/k_s = 2c_s t_{\rm dec}$ have performed exactly one-half oscillation.

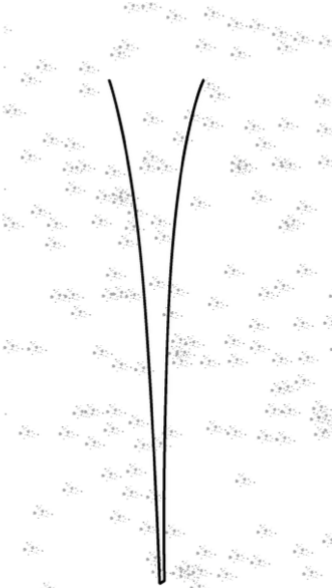


The angle under which the size $L = r_s$ is seen today, depends on the distance of the last scattering surface, $d_A(z_s) = t_0 - t_{\rm dec}$ $\theta_s = r_s/d_A(z_s)$. The angle corresponds to the harmonic $\ell \simeq \pi/\theta_s$.

Within the flat sky approximation a change, $d_A \to d'_A$ changes the CMB power spectrum like (M. Vonlanthen, S. Räsanen and RD, arXiv:1003.0810)

$$\boxed{C'_\ell = \left(\frac{d'_A}{d_A}\right)^2 C_{\frac{d'_A}{d_A}\ell}}$$

Aggregated lensing



On average 0-geodesics diverge due to the matter outside the beam.

This renders angles smaller $\theta \searrow$ and distances $d_A = L/\theta \nearrow$

On average structure mimics negative curvature.

Effects on the distance by large scale clustering

The Friedmann formula

$$(1+z)d_A(z) = \int_0^z H(z)^{-1}dz = \frac{h}{H_0}\int_0^z \frac{1}{\sqrt{\omega_m(1+z)^3+\omega_K(1+z)^2+\omega_x(z)}}dz$$

is strictly valid only in an unperturbed Friedmann universe.

Once matter clusters this formula becomes perturbed,

$$d_A(z) \to d_A'(z) = d_A(1+\langle\Delta(z)\rangle)\,.$$

We take the expectation value to get a direction independent mean value of this change.

Let us assume we can perturb d_A in the form ($\chi(z) = t_0 - t(z)$),

$$d_A(z,\mathbf{n}) = \frac{1}{1+z}\left[\chi(z) + d_A^{(1)}(z,\mathbf{n}) + \frac{1}{2}d_A^{(2)}(z,\mathbf{n}) + \cdots\right]$$

where to first order $\left\langle d_A^{(1)}(z,\mathbf{n})\right\rangle = 0$.

Effects on the distance by large scale clustering: Flux conservation

General relativity (photon number conservation) implies that for an arbitrary source the total flux

$$F = \left\langle \frac{\mathcal{L}}{4\pi d_L^2} \right\rangle$$

remains unchanged is we rearrange the matter in the ball around the source. With $d_L = (1+z)^2 d_A$ we therefore expect

$$\langle d_A(z, \mathbf{n})^{-2} \rangle = (\bar{d}_A)^{-2}$$

Effects on the distance by large scale clustering: Flux conservation

General relativity (photon number conservation) implies that for an arbitrary source the total flux

$$F = \left\langle \frac{\mathcal{L}}{4\pi d_L^2} \right\rangle$$

remains unchanged is we rearrange the matter in the ball around the source. With $d_L = (1+z)^2 d_A$ we therefore expect

$$\langle d_A(z,\mathbf{n})^{-2} \rangle = (\bar{d}_A)^{-2}$$

but

$$d_A(z,\mathbf{n})^{-2} = (\bar{d}_A)^{-2} \left[1 - 2\frac{d_A^{(1)}(z,\mathbf{n})}{\chi(z)} - \frac{d_A^{(2)}(z,\mathbf{n})}{\chi(z)} + 3\left(\frac{d_A^{(1)}(z,\mathbf{n})}{\chi(z)} \right)^2 + \cdots \right] .$$

Taking expectation values on both sides yields

$$\boxed{\langle d_A^{(2)} \rangle = 3\frac{\left\langle \left(d_A^{(1)} \right)^2 \right\rangle}{\chi} .}$$

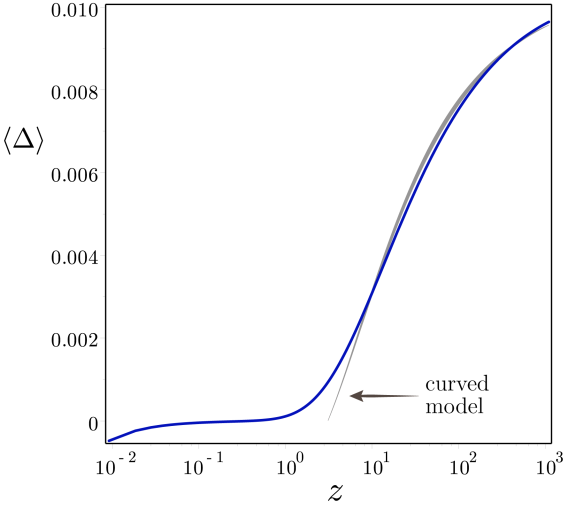
Effects on the mean distance by large scale clustering $\Delta(z)$

$d_A^{(1)}$ has been calculated in linear perturbation theory
(see, e.g. C. Bonvin, RD, A. Gasparini, arXiv:astro-ph/0511183).
The dominant term for large redshifts is the lensing term given by

$$\frac{d_A^{(1)}}{\bar{d}_A}(z,\mathbf{n}) = \int_0^{\chi(z)} \frac{d\chi}{\chi}\frac{\chi(z)-\chi}{\chi(z)}\Delta_2\Phi(t(\chi),\mathbf{n}\chi)\,.$$

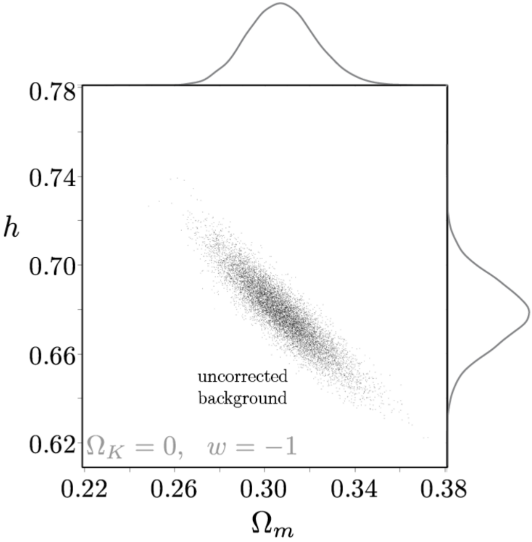
$$\begin{aligned}
\langle d_A\rangle(z) &= \bar{d}_A(z)[1+\Delta(z)],\\
\Delta &= \frac{3}{2}\left\langle\left(\frac{d_A^{(1)}}{\chi(z)}\right)^2\right\rangle = 6\pi\sum_{\ell=0}^{\infty}\left[\frac{\ell(\ell+1)}{2\ell+1}\right]^2\int_0^{\chi(z)}\frac{d\chi}{\chi}\frac{(\chi(z)-\chi)^2}{\chi(z)^2}\mathcal{P}_\Phi\Big|_{k=(\ell+1/2)/\chi}\\
&\propto (k_{\rm eq}\chi(z))^3\ .
\end{aligned}$$

Effects on the mean distance by large scale clustering $\Delta(z)$



(ΛCDM with $\omega_m = 0.14, h = 0.68, \omega_b = 0.0222$ and $n_s = 0.96$).
Open fit: $\Omega_K = 0.0066$.

Effect on the Hubble parameter



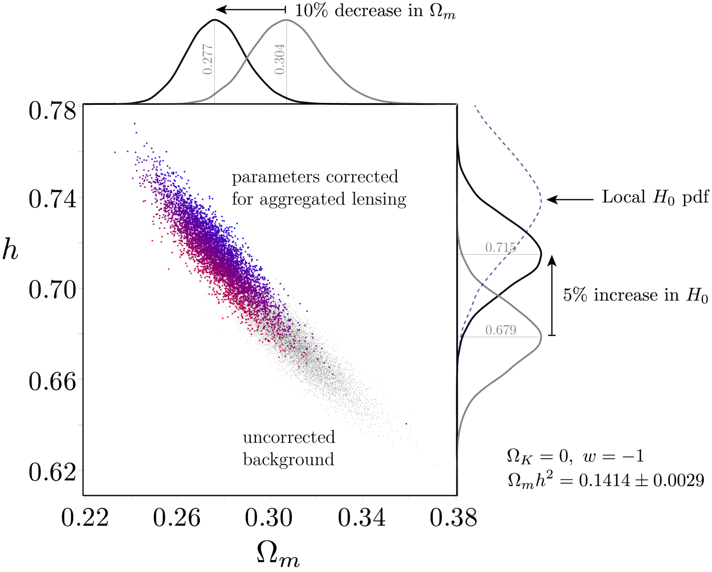
$100\theta_s = 1.04131 \pm 0.00062$

$r_s = (144.71 \pm 0.60)\mathrm{Mpc}$

$d_A = (13897 \pm 58)\mathrm{Mpc}$

$\omega_m = 0.14 \pm 0.0029$

Effect on the Hubble parameter



Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.
- The bulk of the signal comes from relatively small scales, from $k_{\rm eq} \sim 0.01\,h{\rm Mpc}^{-1}$ to $k_{\rm max} \simeq 1\,h{\rm Mpc}^{-1}$ and large redshifts $z \in [10, 300]$. But scales down to 10kpc still contribute.

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.
- The bulk of the signal comes from relatively small scales, from $k_{\rm eq} \sim 0.01\,h\text{Mpc}^{-1}$ to $k_{\max} \simeq 1\,h\text{Mpc}^{-1}$ and large redshifts $z \in [10, 300]$. But scales down to 10kpc still contribute.
- Our calculation is an approximation as it takes into account only $\langle\Delta\rangle$.

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.
- The bulk of the signal comes from relatively small scales, from $k_{\rm eq} \sim 0.01\,h\mathrm{Mpc}^{-1}$ to $k_{\max} \simeq 1\,h\mathrm{Mpc}^{-1}$ and large redshifts $z \in [10, 300]$. But scales down to 10kpc still contribute.
- Our calculation is an approximation as it takes into account only $\langle\Delta\rangle$.
- The true distance depends on direction, $d_A = \bar{d}_A[1 + \Delta(\mathbf{n})]$ and

$$\left(\frac{\Delta T}{T}\right)(\mathbf{n}) = \left(\frac{\Delta T}{T}\right)(\mathbf{x} = d_A(\mathbf{n})\mathbf{n}, z_s)$$

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.
- The bulk of the signal comes from relatively small scales, from $k_{\rm eq} \sim 0.01 h\mathrm{Mpc}^{-1}$ to $k_{\max} \simeq 1 h\mathrm{Mpc}^{-1}$ and large redshifts $z \in [10, 300]$. But scales down to 10kpc still contribute.
- Our calculation is an approximation as it takes into account only $\langle\Delta\rangle$.
- The true distance depends on direction, $d_A = \bar{d}_A[1 + \Delta(\mathbf{n})]$ and

$$\left(\frac{\Delta T}{T}\right)(\mathbf{n}) = \left(\frac{\Delta T}{T}\right)(\mathbf{x} = d_A(\mathbf{n})\mathbf{n}, z_s)$$

- There are integrated contributions...

Conclusion and Outlook

- When Fluctuations are taken into account the distance to the CMB increases.
- From the observed distance we therefore infer a smaller background distance and hence a larger Hubble parameter. This completely removes the tension with local measurements.
- The bulk of the signal comes from relatively small scales, from $k_{\rm eq} \sim 0.01\,h\mathrm{Mpc}^{-1}$ to $k_{\max} \simeq 1\,h\mathrm{Mpc}^{-1}$ and large redshifts $z \in [10, 300]$. But scales down to 10kpc still contribute.
- Our calculation is an approximation as it takes into account only $\langle\Delta\rangle$.
- The true distance depends on direction, $d_A = \bar{d}_A[1 + \Delta(\mathbf{n})]$ and

$$\left(\frac{\Delta T}{T}\right)(\mathbf{n}) = \left(\frac{\Delta T}{T}\right)(\mathbf{x} = d_A(\mathbf{n})\mathbf{n}, z_s)$$

- There are integrated contributions...
- Nongaussianities...

Another derivation of $\langle\Delta\rangle$

Lens map:

$$D : (\theta, \varphi) \mapsto \mathbf{x} \qquad (D_{ab}) = \lambda \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} ,$$

$\kappa =$ convergence, $\gamma = \gamma_1 + i\gamma_2 =$ complex shear.

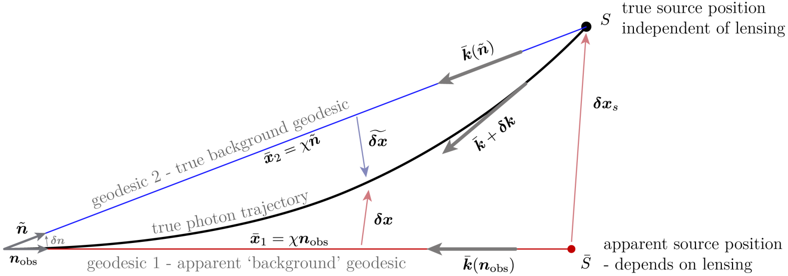
Another derivation of $\langle\Delta\rangle$

Lens map:

$$D : (\theta, \varphi) \mapsto \mathbf{x} \qquad (D_{ab}) = \lambda \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} ,$$

$\kappa =$ convergence, $\gamma = \gamma_1 + i\gamma_2 =$ complex shear.
We perturb all quantities to 2nd order

$$\begin{aligned} \lambda &= \chi + \lambda^{(1)} + \frac{1}{2}\lambda^{(2)} \qquad \text{(perts. subdominant)} \\ \kappa &= \kappa^{(1)}(\bar{S}) + \frac{1}{2}\kappa^{(2)}(\bar{S}) \\ \gamma_{1,2} &= \gamma_{1,2}^{(1)}(\bar{S}) + \cdots \end{aligned}$$



Another derivation of $\langle\Delta\rangle$

$$\begin{aligned} d_A^2(S) &= \det(D_{ab}) = \chi^2\left[1 - 2\kappa^{(1)}(\bar S) - \kappa^{(2)}(\bar S) + (\kappa^{(1)})^2(\bar S) - (\gamma_1^{(1)})^2(\bar S) - (\gamma_2^{(1)})^2(\bar S)\right] \\ &= \chi^2\left[1 - 2\kappa^{(1)}(\bar S) + \text{total divergence}\right], \end{aligned}$$

(F. Bernardeau, C. Bonvin & F. Vernizzi, arXiv:1112:4430)

Another derivation of $\langle\Delta\rangle$

$$
\begin{aligned}
d_A^2(S) &= \det(D_{ab}) = \chi^2\left[1 - 2\kappa^{(1)}(\bar{S}) - \kappa^{(2)}(\bar{S}) + (\kappa^{(1)})^2(\bar{S}) - (\gamma_1^{(1)})^2(\bar{S}) - (\gamma_2^{(1)})^2(\bar{S})\right] \\
&= \chi^2\left[1 - 2\kappa^{(1)}(\bar{S}) + \text{total divergence}\right],
\end{aligned}
$$

(F. Bernardeau, C. Bonvin & F. Vernizzi, arXiv:1112:4430)
Square root:

$$
d_A(S) = \chi\left[1 - \kappa^{(1)}(\bar{S}) - \frac{1}{2}\left(\kappa^{(1)}(\bar{S})\right)^2 + \text{total divergence}\right].
$$

Another derivation of $\langle\Delta\rangle$

$$\begin{aligned} d_A^2(S) &= \det(D_{ab}) = \chi^2\left[1-2\kappa^{(1)}(\bar S)-\kappa^{(2)}(\bar S)+(\kappa^{(1)})^2(\bar S)-(\gamma_1^{(1)})^2(\bar S)-(\gamma_2^{(1)})^2(\bar S)\right] \\ &= \chi^2\left[1-2\kappa^{(1)}(\bar S)+\text{total divergence}\right], \end{aligned}$$

(F. Bernardeau, C. Bonvin & F. Vernizzi, arXiv:1112:4430)
Square root:

$$d_A(S)=\chi\left[1-\kappa^{(1)}(\bar S)-\frac{1}{2}\left(\kappa^{(1)}(\bar S)\right)^2+\text{total divergence}\right].$$

We want to determine the 2nd order part of this quantity.
For this we subtract the 0th and 1st order. But to subtract the 1st order we have to evaluate both sides at the same position. To 2nd order this does make a difference.

$$d_A(S)=\chi\left[1+\frac{d_A^{(1)}(S)}{\chi}+\frac{1}{2}\frac{d_A^{(2)}(S)}{\chi}\right]$$

Another derivation of $\langle\Delta\rangle$

Expanding also

$$\begin{aligned}\kappa^{(1)}(\bar{S}) &= \kappa^{(1)}(S) - \delta x^a \partial_a \kappa^{(1)}(\bar{S}) = \kappa^{(1)}(S) + (\partial_a \delta x^a)\kappa^{(1)}(\bar{S}) \\ &= \kappa^{(1)}(S) - 2[\kappa^{(1)}(\bar{S})]^2 + \text{total divergence}\end{aligned}$$

Another derivation of $\langle\Delta\rangle$

Expanding also

$$\begin{aligned}\kappa^{(1)}(\bar{S}) &= \kappa^{(1)}(S) - \delta x^a \partial_a \kappa^{(1)}(\bar{S}) = \kappa^{(1)}(S) + (\partial_a \delta x^a)\kappa^{(1)}(\bar{S}) \\ &= \kappa^{(1)}(S) - 2[\kappa^{(1)}(\bar{S})]^2 + \text{total divergence}\end{aligned}$$

(To first order $(\partial_a \delta x^a) = -2\kappa^{(1)}$.) With the first order identity

$$-\kappa^{(1)}(S) = \frac{\delta d_A(S)}{\chi}$$

we finally obtain

$$\begin{aligned}\frac{d_A^{(2)}(S)}{\chi} &= 3(\kappa^{(1)})^2(\bar{S}) + \text{ total divergence} \\ &= 3\frac{(d_A^{(1)})^2}{\chi^2} + \text{ total divergence.}\end{aligned}$$