The Effective Action– what it is, what it isn't.

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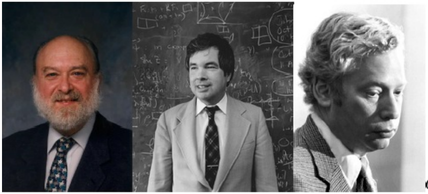
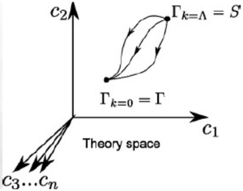
The effective action: theory space





Effective field theory represents the pinnacle of our understanding of what quantum field theories really are. Central object of interest is the effective action [1]. Ingredients are:

- Symmetries.
- Degrees of freedom on which these are represented[1].
- A desired accuracy (order) to which we would like to compute.
- e.g. Lorentz invariant theory w/ shift symmetric scalar $\mathcal{L}_{\rm eff}[\varphi] = -\frac{1}{2}(\partial\varphi)^2 + \frac{c_1}{\Lambda^4}(\partial\varphi)^4 + \frac{c_2}{\Lambda^8}(\partial\varphi)^6 + \frac{c_3}{\Lambda^8}(\partial\varphi)^2\Box(\partial\varphi)^2 + \ldots$
- The $\{c_i\}$ are the so called Wilson coefficients, fixed by a *finite* number of observations at a particular scale μ .

[1] Either linearly or non-linearly

The effective action: theory space

The effective action is not a fundamental object. It is merely a *bootstrap* which allows us to compute physical *observables*. These could be

- S-matrix elements[2] between (asymptotic) physical states $\rightarrow$ amplitudes, cross sections...
- (finite time) correlation functions.
- ... Sometimes the only way to define a theory.
- *If* you started with knowledge of the UV theory, in principle:
$$e^{iS_{\rm eff}(\ell,\Lambda)} := \int \mathcal{D}h_\Lambda\, e^{iS(\ell,h)}$$
- In practice, we identify the low energy symmetries and d.o.f.'s of the theory, and write down *all* consistent operators...
- Any particular EFT is a point in 'theory space' (the space spanned by a *basis* of such operators).
- Classify them according to their operator 'dimension' Δ in d spacetime dimensions:
- Relevant operators: $\Delta < d$
  Marginal operators: $\Delta = d$
  Irrelevant operators: $\Delta > d$

[2] Does not always exist

Relevant and irrelevant operators

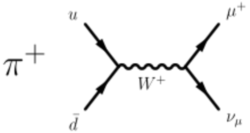
Why this terminology? Consider first, the action for a free scalar field theory in d-dimensions [2]:

- $S = -\frac{1}{2}\int d^dx \left[(\partial\varphi)^2 + m^2\varphi^2\right]$
- $[...] :=$ mass dimension of a particular quantity (natural units):
  $$[\varphi] = (d-2)/2;\ [m^2] = 2$$
- Objects of interest: $G_n(x_1, ..., x_n) := \langle\varphi(x_1)...\varphi(x_n)\rangle_S$
- Under the rescaling $x = \lambda x', \varphi(x) = \lambda^{(2-d)/2}\varphi'(x')$
  $$S' = -\frac{1}{2}\int d^dx' \left[(\partial'\varphi')^2 + m^2\lambda^2\varphi'^2\right]$$
- Correlation functions rescale as
  $$\langle\varphi(\lambda x_1)...\varphi(\lambda x_n)\rangle_S = \lambda^{n(2-d)/2}\langle\varphi'(x_1)...\varphi'(x_n)\rangle_{S'}$$
- Adding interactions $g_4\frac{\varphi^4}{4!} + g_6\frac{\varphi^6}{6!};\quad [g_4] = 0,\ [g_6] = -2$
  $$S'_{\Lambda'} = -\int d^dx' \left[\frac{1}{2}(\partial'\varphi')^2 + \frac{m^2\lambda^2}{2}\varphi'^2 + g_4\frac{\varphi'^4}{4!} + g_6\frac{\varphi'^6}{\lambda^2 6!}\right]$$
- N.B. Implicitly, cutoff $\Lambda' = \lambda\Lambda$. See that in the limit $\lambda \to \infty$, φ^6 term vanishes as $1/\lambda^2$, hence its 'irrelevance'.
- Integrating out modes w/ $\frac{\Lambda}{\lambda} < p < \Lambda$ restores cutoff back to Λ – results in modifications to *all* couplings in theory space previously defined for S at Λ : RG flow (the hard part).

Relevant and irrelevant operators

Typically, we are interested in computing observables at low energies far away from the scale at which ‘new physics’ becomes relevant and where the effective description breaks down [2].

- However, low energy does not mean zero energy.
- e.g. consider pion dynamics at $1GeV$. N.B. $M_W \sim 80GeV$ – in the parent theory:



- Therefore the difference in scales $\lambda = 80$.
- φ^6 operator gives $1/\lambda^2 \sim 1/80^2$ corrections,$\varphi^8 \rightarrow 1/\lambda^4 \sim 1/80^4$

The effective prescription

Therefore before diving into the full formalism, from dimensional analysis alone we can identify the outline of the prescription one has to follow:

- From the free part of the action, determine the canonical dimension of the field.
- Determine the mass dimension of all couplings.
- Coupling constants of dimension δ scale as λ^{δ} .
- Relevant/ marginal/ irrelevant operators have couplings with δ less than/ equal to/ greater than d , the number of spacetime dimensions.
- Accuracy up to order $1/\lambda^{p}$ requires us to include all operators w/ dimension $\Delta \leq d + p$, i.e. with couplings w/ $\delta \geq -p$.

The renormalization group

Recall that the main objects of interest are n-point correlation functions:

- $G_n(x_1, ..., x_n) := \langle \varphi(x_1)...\varphi(x_n)\rangle_S$

- Consider an interacting theory regulated by a cutoff Λ
$$S_\Lambda = -\int d^dx \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 + g_4\frac{\varphi^4}{4!} + g_6\frac{\varphi^6}{6!}\right]$$
Under co-ordinate rescaling $x = \lambda x'$
$$S'_{\Lambda'} = -\int d^dx' \left[\frac{1}{2}(\partial'\varphi')^2 + \frac{m^2\lambda^2}{2}\varphi'^2 + g_4\frac{\varphi'^4}{4!} + g_6\frac{\varphi'^6}{\lambda^2 6!}\right]$$
N.B. cutoff also rescaled to $\Lambda' = \lambda\Lambda$.
- Correlation functions rescale as
$$\langle\varphi(\lambda x_1)...\varphi(\lambda x_n)\rangle_{S,\Lambda} = \lambda^{n(2-d)/2}\langle\varphi'(x_1)...\varphi'(x_n)\rangle_{S',\Lambda'}$$
$$G_n(\{\lambda x\}; m^2, g_4, g_6; \Lambda) = \lambda^{n(2-d)/2}G_n(\{x\}; \lambda^2m^2, g_4, g_6\lambda^{-2}; \lambda\Lambda)$$
- LHS is the desired correlation function, to understand its IR behaviour, need to compute RHS for $\lambda\Lambda \to \Lambda$.
- Doing so, has effectively 'coarse grained' the system in momentum space (Wilson)– the analogue of Kadanoff's position space (block spin) coarse graining.

The renormalization group

We know that no physical quantity can depend on Λ (more physically, the regularization scheme).

- Therefore:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial c_i} + n\gamma_\varphi\right] G_n = 0$$

- ... where the $\{c_i\}$ are the couplings of the theory, and where we have merely applied the chain rule–

$$\beta_j(\{c_i\},\Lambda) := \Lambda\frac{\partial c_j}{\partial\Lambda};\ \gamma_\varphi(\{c_i\},\Lambda) = \Lambda\frac{\partial\varphi}{\partial\Lambda}$$

- γ_φ is the so called 'anomalous dimension' of the field φ , and arises from wave-function renormalization.
- Solving these coupled non-linear PDE's– the 'renormalization group equations' is the hard labour of perturbative QFT!
- Formally, given $\{c_j(\Lambda)\}$ that solve $\beta_j = \Lambda\frac{\partial c_j}{\partial\Lambda}$

$$G_n(\{x\}, c_i(\Lambda_1), \Lambda_1) = e^{-n\int_{\Lambda_1}^{\Lambda_2}\gamma_\varphi(\Lambda)d\log\Lambda} G_n(\{x\}, c_i(\Lambda_2), \Lambda_2)$$

The renormalization group

Disclaimer:

Although the present discussion is organized around Wilson's original idea of a hard momentum space cutoff Λ , in practice it is much more useful to work with a regularization scheme that preserves as many symmetries of the original Lagrangian as possible (gauge, diff, etc). For these and other reasons (that will become clear later) we will subsequently use 'dimensional regularization'.

The renormalization group

There are privileged points in theory space where $\beta_j[c_i(\Lambda)] = 0$; the so called RG fixed points.

- These can either be stable, or unstable and exist either in the UV or the IR.
- QFT in its most general terms, is the study of RG flows– how theories evolve from the UV to the IR [3].

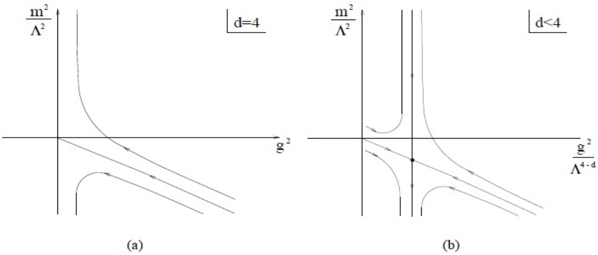


Figure 2: Renormalization group flows for the Ising scalar field theory in (a) $d = 4$ and (b) $d < 4$.

(Figure from Gubser and Sondhi; arXiv:hep-th/0006119)

The renormalization group

- Recalling the scaling relation
$$G_n(\{\lambda x\}; c_i; \Lambda) = \lambda^{n(2-d)/2} G_n(\{x\}; \lambda^{\delta_i} c_i(\Lambda); \lambda\Lambda)$$
together with the formal solution
$$G_n(\{x\}, c_i(\Lambda_1), \Lambda_1) = e^{-n\int_{\Lambda_1}^{\Lambda_2}\gamma_\varphi(\Lambda) d\log\Lambda} G_n(\{x\}, c_i(\Lambda_2), \Lambda_2) \to$$
$$G_n(\{\lambda x\}, c_i(\Lambda), \Lambda) = \lambda^{n(2-d)/2} e^{-n\int_{\Lambda}^{\Lambda/\lambda}\gamma_\varphi(\tilde\Lambda) d\log\tilde\Lambda} G_n(\{x\}, \lambda^{\delta_i} c_i(\Lambda/\lambda), \Lambda)$$
- Imagine a theory w/ a dimensionless coupling g for which $\exists$ fixed point: $\beta(g_*) = 0$: e.g. $SU(N_c)$ YM w/ N_f flavours
$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} + \beta_1 \frac{g^5}{(16\pi^2)^2} + \ldots$$
$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f; \;\; \beta_1 \sim \mathcal{O}(N_c^2, N_c N_f)$$
- Fixed point at $g_*^2 = 16\pi^2 \frac{\beta_0}{\beta_1}$
- For all other operators (e.g. four-Fermion contact interaction)
$$\Lambda \frac{\partial c_i}{\partial \Lambda} = \widetilde{\gamma}_i(g) \frac{g^2(\Lambda)}{16\pi^2} c_i(\Lambda) = \widetilde{\gamma}_i^* \frac{g_*^2}{16\pi^2} c_i(\Lambda) \to \frac{c_i(\Lambda_1)}{c_i(\Lambda_2)} = \left(\frac{\Lambda_1}{\Lambda_2}\right)^{\widetilde{\gamma}_i^*}$$
$$\exp\left(-\int_{\Lambda_2}^{\Lambda_1} \gamma_\varphi^* d\log\Lambda\right) = \left(\frac{\Lambda_1}{\Lambda_2}\right)^{-\gamma^*}$$
- So that:
$$G_n(\{\lambda x\}, c_i(\Lambda), \Lambda) = \lambda^{n(2-d)/2} \lambda^{n\gamma_\varphi^*} G_n\left(\{x\}, \lambda^{\delta_i - \widetilde{\gamma}_i^*} c_i(\Lambda/\lambda), \Lambda\right)$$

RG fixed points + EFT prescription

This equation shows scale invariance at the IR fixed point:

- $G_n\left(\{\lambda x\}, c_i(\Lambda), \Lambda\right) = \lambda^{n(2-d)/2}\lambda^{n\gamma^*_\varphi} G_n\left(\{x\}, \lambda^{\delta_i - \tilde{\gamma}^*_i} c_i(\Lambda/\lambda), \Lambda\right)$
- ... but with quantum operators with dimensions that differ from their classical scaling dimensions.
- Operators now acquire an 'anomalous dimension' $\Delta_i = d - \Delta^0_i + \gamma^*_i$
- Weakly coupled QFT's: quantum corrections can at most turn marginal operators into relevant, or irrelevant operators e.g. triviality of $\lambda\varphi^4$ in D=4.
- Must work from the outset with a good set of variables to describe dynamics s.t. anomalous dimensions are small...
- Otherwise one loses track of what the true d.'s.o.f. are!

Classical vs quantum degrees of freedom

Anomalous dimensions could be large. Classical degrees of freedom, as identified by the classical action need not bear any resemblance to the true (e.g. weakly coupled) degrees of freedom of the quantum theory.

- e.g. Pions– a composite operator of the form $[\bar{\psi}\psi] = 3$ is expected to behave as a scalar ($[\varphi] = 1$) at low energies. i.e. $\gamma^* = -2$.
- i.e. pions from QCD not practical. However, pions from the chiral Lagrangian is fine (it's a weakly coupled theory). Moral: pick your d.o.f.'s wisely!
- Q) what do the following theories in 2d have to do with each other?
  $\mathcal{L} = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{\alpha}{\beta^2}\left(cos(\beta\theta) - 1\right)$ (sine-Gordon)
  $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$ (Thirring)
- A) They're the same theory!

Classical vs quantum degrees of freedom

$\mathcal{L} = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{\alpha}{\beta^2}\left(cos(\beta\theta) - 1\right)$ (sine-Gordon)
$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$ (Thirring)

- sine-Gordon model has soliton solutions w/ [4] $E_{sol} = \frac{8\sqrt{\alpha}}{\beta^2}$.
- Solitons annihilate w/ antisolitons (w.r.t. topological charge)
- Multi-soliton solutions exhibit Pauli exclusion.
- cf. [4] Conjecture (Skyrme, 1961): Thirring fermions $\equiv$ Sine-Gordon solitons.
- Coleman (1975): operator equivalence of

$$Zm\bar{\psi}\frac{1\mp\gamma^5}{2}\psi \leftrightarrow -\frac{\alpha}{\beta^2}e^{\pm i\theta}$$

  provided we identify

$$\frac{\beta^2}{4\pi} = \frac{1}{1+g/\pi}$$

- Phenomenon known as (Abelian) bosonization.
- Strongly coupled sine-Gordon model w/ $\beta^2 \approx 4\pi$ maps onto weakly coupled Thirring model w/ $g \approx 0$. Either work with strongly interacting bosonic theory w/ large anomalous dimensions, or weakly coupled fermionic theory w/ small anomalous dimensions.

Picking the right operator basis

Related to the issue of identifying the right degrees of freedom to describe the dynamics is the issue of operator *basis*– a set of independent operators whose linear combinations span theory space.

- Field redefinitions renders certain operators 'redundant'.
- Field redefinitions $\leftrightarrow$ coordinate transformations.
- Underpinning this is the S-matrix 'Equivalence Theorem'.
- Any invertible field redefinition gives identical on shell S-matrix elements, even if general correlation functions will be different.
- Also generalizes to finite time 'in-in' matrix elements.
- Proved using the LSZ formula. Simple proof at tree-level presented by Coleman, Wess and Zumino [5].

The Equivalence Theorem

Consider a Lagrangian of the form:

$$\mathcal{L}[\varphi] = \mathcal{L}_0[\varphi] + \mathcal{L}_1[\varphi]$$

- with $\mathcal{L}_0[\varphi]$ the free part that defines the propagator and $\mathcal{L}_1[\varphi]$ containing all other operators.
- Consider φ a non-linear but local function of some other variable χ , s.t. $\varphi = \chi F[\chi], \;\; F[0] = 1$.
- In terms of χ , we again make the separation

  $$\mathcal{L}[\chi F[\chi]] := \mathcal{L}_0[\chi] + \mathcal{L}_2[\chi]$$

  N.B. $\mathcal{L}_0$ is the *same* function in both cases.
- Now compare correlation functions computed for the original Lagrangian with those of $\mathcal{L}'[\varphi] = \mathcal{L}_0[\varphi] + \mathcal{L}_2[\varphi]$
- This new theory has the same propagators, but completely different interactions.
- Equivalence Theorem: on-shell S-matrix elements computed for $\mathcal{L}$ are the same as those for $\mathcal{L}'$

The Equivalence Theorem

- Proof [5]: Given any Lagrangian $\mathcal{L}[\varphi]$, define:
$$\mathcal{L}[\varphi, a] := a^{-2}\mathcal{L}[a\varphi]$$
- Consider a connected Feynman diagram with $E/I/V$ external/ internal lines/ vertices, respectively.
- Each vertex w/ N_i lines attached to it has $N_i - 2$ powers of a . Hence, each diagram carries P powers of a defined as
$$P = \sum_{i=1}^{V}(N_i - 2)$$
- However a line is either internal (connecting two vertices) or external, so $\sum_{i=1}^{V} N_i = E + 2I$
- So that $P = E + 2I - 2V$
- However, the number of loops L satisfies $L = I - V + 1$, so that $P = E + 2L - 2 = E - 2$ at tree level.
- Now using $\varphi = \chi F[a\chi]$, we have
$$\mathcal{L}[\chi F[a\chi], a] = a^{-2}\mathcal{L}[a\chi F[a\chi]]$$
- Connection between power of a and the number of lines attached to each vertex the same. S-matrices calculated from the two Lagrangians using exact solutions are equal, therefore all coefficients of powers of a must be equal too.

Redundant operators

One can use this freedom to field redefine to eliminate many operators otherwise allowed by the symmetries of the problem [1].

- Recall Lorentz invariant theory w/ shift symmetric scalar:
  $$\mathcal{L}_{\rm eff}[\varphi] = -\frac{1}{2}(\partial\varphi)^2 + \frac{c_1}{\Lambda^4}(\partial\varphi)^4 + \frac{c_2}{\Lambda^8}(\partial\varphi)^6 + \frac{c_3}{\Lambda^8}(\partial\varphi)^2\Box(\partial\varphi)^2 + \ldots$$
- What about $\frac{\tilde{c}}{\Lambda^2}\Box\varphi\Box\varphi$ or $\frac{\tilde{c}}{\Lambda^2}\partial_\mu\varphi\Box\partial^\mu\varphi$ terms?
- Clearly in flat space[3], they differ by an integration by parts.
- Under field redefinition $\varphi \to \varphi + \frac{\kappa}{\Lambda^2}\Box\varphi$, variation of the quadratic part of the action gives
  $$\Delta\mathcal{L} = \frac{\kappa}{\Lambda^2}\Box\varphi\Box\varphi$$
- Setting $\kappa = -\tilde{c}$ allows us to eliminate the fist term.
- Higher order terms modified. Making subsequent field redefinitions with higher powers of φ do not affect lower order terms with redundant operators eliminated.
- Can proceed order by order to eliminate all redundant operators to obtain an *operator basis*, whose coefficients parametrize *theory space*.

[3] argument generalizes straightforwardly to curved backgrounds

Non-linearly realized symmetries

When we want to write down a basis of operators consistent with the symmetries of the problem, one could either realize the symmetries *linearly*, or *non-linearly* [6].

- e.g. consider an N dimensional vector field ϕ^a (w/ a flat field space metric) s.t. $\phi^a\phi_a = 1$. A theory with linearly realized $O(N)$ invariance is given by [7]:

  $$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi^a\partial^\mu\phi_a + \lambda\left(\phi^a\phi_a - 1\right)$$

  with λ , a Lagrange multiplier enforcing $\phi^a\phi_a = 1$.
- Consider now the fact that any vector ϕ^a can be represented as

  $$\phi^a(x) = G^a_b(x)u^b$$

  where e.g. $u^a = (1, 0, ..., 0)$ is some fixed vector of norm one, and $G^a_b(x)$ is a linear representation of any element of $SO(N)$.
- Little group of u is $SO(N-1)$. Demonstrates isomorphism between the coset space $O(N)/O(N-1)$ and S_{N-1} .

Non-linear realizations of symmetries

But what if we only wanted to write down independent degrees of freedom?

- Writing $$\phi^a(x) = (\sigma(x), \pi^\alpha(x))$$ with $\pi^\alpha(x)$ an $N-1$ dimensional vector.
- The condition $\phi^a\phi_a = 1$ implies $$\sigma(x) = (1-\pi^\alpha\pi_\alpha)^{1/2}$$
- Decompose the generators of $O(N)$ into the generators of $O(N-1)$ and the complimentary set. The generators of $O(N-1)$ act *linearly* on π^α . The complimentary generators act as $$\delta\pi^\alpha = \omega^\alpha(1-\pi^2(x))^{1/2}$$ $$\delta\sigma = \delta(1-\pi^2(x))^{1/2} = -\omega^\alpha\pi_\alpha(x)$$ where the ω^a are the infinitesimal parameters of the transformation.
  - $\mathcal{L} = -\frac{1}{2}G_{\alpha\beta}(\pi)\partial_\mu\pi^\alpha\partial^\mu\pi^\beta; \quad G_{ab} = \delta_{\alpha\beta} + \frac{\pi_\alpha\pi_\beta}{1-\pi^2}$
- Symmetry is *non-linearly realized*. Operators of different order mix under this transformation. Can force non-trivial relations between them (e.g. consistency relations for comoving curvature mode $\mathcal{R}$ in FRW cosmology).

The EFT prescription

- Think about the problem at hand. Consider what the most physically relevant d's.o.f. are at E_c the energy scale of interest.
- Figure out how the linear/non-linear realization of the symmetry of the theory represents on operators.
- Pick a basis of (non-redundant) operators.
- Write down all terms consistent with realization of symmetries in this basis, each term parametrized by an independent Wilson coefficient.
- Decide upon the accuracy to which you want to compute.
- If λ is the ratio Λ/E_c , then for accuracy up to $\mathcal{O}(1/\lambda^k)$, include all operators with dimension $\Delta \leq d + k$.
- Fix the Wilson coefficient for each operator up to $\Delta \leq d + k$ by a finite number of measurements (renormalization conditions) at some scale μ .
- Compute anomalous dimensions, compute RG flow of all couplings of operators w/ $\Delta \leq d + k$ to E_c .

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