CMB anisotropies: Theory and Planck results – supplementary notes

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These notes provide some useful background material for the course on CMB anisotropies, and fill in some gaps in particularly involved calculations. The notes will evolve over the four weeks of the course. Currently, they include the following material:

- Statistics of random fields – Sec. 1
- Cosmological linear perturbation theory – Sec. 2
- Boltzmann equation for Thomson scattering of unpolarized radiation – Sec. 3
- CMB anisotropies from gravitational waves – Sec. 4
- Generation of polarization by Thomson scattering – Sec. 5
- Polarization from scalar perturbations – Sec. 6
- Polarization from gravitational waves – Sec. 7

1 Statistics of random fields

Theory (e.g. quantum mechanics during inflation) only allows us to predict the statistical properties of cosmological fields (such as the matter overdensity $\delta\rho$). Here, we explore the basic statistical properties enforced on such fields by assuming the physics that generates the initial fluctuations, and subsequently processes them, respects the symmetries of the background cosmology, i.e. isotropy and homogeneity.

Throughout, we denote expectation values with angle brackets, e.g. $\langle \delta\rho \rangle$; you should think of this as a quantum expectation value or an average over a classical ensemble[1]. To keep the Fourier analysis simple, we shall only consider flat ($K = 0$) background models and we denote comoving spatial positions by $\mathbf{x}$.

[1] For a recent review on the question of why quantum fluctuations from inflation can be treated as classical, see Keifer & Polarski (2008), available online at http://arxiv.org/abs/0810.0087.

1.1 Random fields in 3D Euclidean space

Consider a random field $f(\mathbf{x})$ – i.e. at each point $f(\mathbf{x})$ is some random number – with zero mean, $\langle f(\mathbf{x})\rangle = 0$. The probability of realising some field configuration is a *functional* $\Pr[f(\mathbf{x})]$. *Correlators* of fields are expectation values of products of fields at different spatial points (and, generally, times). The two point correlator is

$$\xi(\mathbf{x},\mathbf{y}) \equiv \langle f(\mathbf{x})f(\mathbf{y})\rangle = \int \mathcal{D}f\,\Pr[f]f(\mathbf{x})f(\mathbf{y})\,, \tag{1.1}$$

where the integral is a *functional integral* (or path integral) over field configurations.

Statistical homogeneity means that the statistical properties of the translated field,

$$\hat{T}_{\mathbf{a}}f(\mathbf{x}) \equiv f(\mathbf{x}-\mathbf{a})\,, \tag{1.2}$$

are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{T}_{\mathbf{a}}f(\mathbf{x})]$. For the two-point correlation, this means that

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{a}) \quad \forall\mathbf{a} \\ \Rightarrow\quad &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{y})\,, \end{aligned} \tag{1.3}$$

so the two-point correlator only depends on the separation of the two points.

Statistical isotropy mean that the statistical properties of the rotated field,

$$\hat{R}f(\mathbf{x}) \equiv f(\mathsf{R}^{-1}\mathbf{x})\,, \tag{1.4}$$

where R is a rotation matrix, are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{R}f(\mathbf{x})]$. For the two-point correlator, we must have

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathsf{R}^{-1}\mathbf{x},\mathsf{R}^{-1}\mathbf{y}) \quad \forall\mathsf{R}\,. \tag{1.5}$$

Combining statistical homogeneity and isotropy gives

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi\left(\mathsf{R}^{-1}(\mathbf{x}-\mathbf{y})\right) \quad \forall\mathsf{R} \\ \Rightarrow\quad &\xi(\mathbf{x},\mathbf{y}) = \xi(|\mathbf{x}-\mathbf{y}|)\,, \end{aligned} \tag{1.6}$$

so the two-point correlator depends only on the distance between the two points. Note that this holds even if correlating fields at different times, or correlating different fields.

We can repeat these arguments to constrain the form of the correlators in Fourier space. We adopt the symmetric Fourier convention, so that

$$f(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{and} \quad f(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} f(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{1.7}$$

Note that for real fields, $f(\mathbf{k}) = f^*(-\mathbf{k})$. Under translations, the Fourier transform acquires a phase factor:

$$\begin{aligned}\hat{T}_{\mathbf{a}} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}-\mathbf{a}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}'}{(2\pi)^{3/2}} f(\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'} e^{-i\mathbf{k}\cdot\mathbf{a}} \\ &= f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{a}} .\end{aligned} \tag{1.8}$$

Invariance of the two-point correlator in Fourier space is then

$$\begin{aligned} & \langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle = \langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}} \quad \forall \mathbf{a} \\ \Rightarrow \quad & \langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle = F(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') ,\end{aligned} \tag{1.9}$$

for some (real) function $F(\mathbf{k})$. We see that different Fourier modes are *uncorrelated.* Under rotations,

$$\begin{aligned}\hat{R} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i(\mathsf{R}^{-1}\mathbf{k})\cdot(\mathsf{R}^{-1}\mathbf{x})} \\ &= f(\mathsf{R}^{-1}\mathbf{k}) ,\end{aligned} \tag{1.10}$$

so, additionally demanding invariance of the two-point correlator under rotations implies

$$\langle \hat{R} f(\mathbf{k}) [\hat{R} f(\mathbf{k}')]^* \rangle = \langle f(\mathsf{R}^{-1}\mathbf{k}) f^*(\mathsf{R}^{-1}\mathbf{k}') \rangle = F(\mathsf{R}^{-1}\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') = F(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') \quad \forall \mathsf{R} . \tag{1.11}$$

(We have used $\delta(\mathsf{R}^{-1}\mathbf{k}) = \det\mathsf{R}\,\delta(\mathbf{k}) = \delta(\mathbf{k})$ here.) This is only possible if $F(\mathbf{k}) = F(k)$ where $k \equiv |\mathbf{k}|$. We can therefore define the *power spectrum*, $\mathcal{P}_f(k)$, of a homogeneous and isotropic field, $f(\mathbf{x})$, by

$$\langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_f(k) \delta(\mathbf{k}-\mathbf{k}') . \tag{1.12}$$

The normalisation factor $2\pi^2/k^3$ in the definition of the power spectrum is conventional and has the virtue of making $\mathcal{P}_f(k)$ dimensionless if $f(\mathbf{x})$ is.

The correlation function is the Fourier transform of the power spectrum:

$$\begin{aligned}\langle f(\mathbf{x}) f(\mathbf{y}) \rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \underbrace{\langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta(\mathbf{k}-\mathbf{k}')} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{y}} \\ &= \frac{1}{4\pi} \int \frac{dk}{k} \mathcal{P}_f(k) \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} .\end{aligned} \tag{1.13}$$

We can evaluate the angular integral by taking $\mathbf{x}-\mathbf{y}$ along the z-axis in Fourier space. Setting $\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) = k|\mathbf{x}-\mathbf{y}|\mu$, the integral reduces to

$$2\pi \int_{-1}^{1} d\mu \, e^{ik|\mathbf{x}-\mathbf{y}|\mu} = 4\pi j_0(k|\mathbf{x}-\mathbf{y}|)\,, \tag{1.14}$$

where $j_0(x) = \sin(x)/x$ is a spherical Bessel function of order zero. It follows that

$$\xi(\mathbf{x},\mathbf{y}) = \int \frac{dk}{k}\, \mathcal{P}_f(k) j_0(k|\mathbf{x}-\mathbf{y}|)\,. \tag{1.15}$$

Note that this only depends on $|\mathbf{x}-\mathbf{y}|$ as required by Eq. (1.6).

The variance of the field is $\xi(0) = \int d\ln k\, \mathcal{P}_f(k)$. A *scale-invariant* spectrum has $\mathcal{P}(k) = \text{const.}$ and its variance receives equal contributions from every decade in k.

1.2 Gaussian random fields

For a Gaussian (homogeneous and isotropic) random field, $\Pr[f(\mathbf{x})]$ is a Gaussian functional of $f(\mathbf{x})$. If we think of discretising the field in N pixels, so it is represented by a N-dimensional vector $\mathbf{f} = [f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_N)]^T$, the probability density function for $\mathbf{f}$ is a multi-variate Gaussian fully specified by the correlation function

$$\langle f_i f_j \rangle = \xi(|\mathbf{x}_i - \mathbf{x}_j|) \equiv \xi_{ij}\,, \tag{1.16}$$

where $f_i \equiv f(\mathbf{x}_i)$, so that

$$\Pr(\mathbf{f}) \propto \frac{e^{-f_i \xi^{-1}_{ij} f_j/2}}{\sqrt{\det(\xi_{ij})}}\,. \tag{1.17}$$

Since $f(\mathbf{k})$ is linear in $f(\mathbf{x})$, the probability distribution for $f(\mathbf{k})$ is also a multi-variate Gaussian. Since different Fourier modes are uncorrelated (see Eq. 1.9), they are statistically *independent* for Gaussian fields.

Inflation predicts fluctuations that are very nearly Gaussian and this property is preserved by *linear* evolution. The cosmic microwave background probes fluctuations mostly in the linear regime and so the fluctuations look very Gaussian. Non-linear structure formation at late times destroys Gaussianity and gives rise to the filamentary cosmic web. Searching for primordial non-Gaussianity to probe departures from simple inflation is a very hot topic but no convincing evidence for primordial non-Gaussianity has yet been found.

1.3 Random fields on the sphere

In cosmology, what we observe is generally a projection of some 3D random field onto the celestial sphere. For example, the CMB anisotropies are mostly the projection of the photon density, bulk velocity and the gravitational potential over the surface of last-scattering. In this section, we shall develop some of the language used to describe random fields on the sphere.

Spherical harmonics form a basis for (square-integrable) functions on the sphere:

$$f(\hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\hat{\mathbf{n}}) \,. \tag{1.18}$$

The Y_{lm} are familiar from quantum mechanics as the position-space representation of the eigenstates of $\hat{L}^2 = -\nabla^2$ and $\hat{L}_z = -i\partial_\phi$:

$$\begin{aligned} \nabla^2 Y_{lm} &= -l(l+1) Y_{lm} \\ \partial_\phi Y_{lm} &= im Y_{lm} \,, \end{aligned} \tag{1.19}$$

with l an integer ≥ 0 and m an integer with $|m| \leq l$. The spherical harmonics are orthonormal over the sphere,

$$\int d\hat{\mathbf{n}}\, Y_{lm}(\hat{\mathbf{n}}) Y^*_{l'm'}(\hat{\mathbf{n}}) = \delta_{ll'} \delta_{mm'} \,, \tag{1.20}$$

so that the *spherical multipole coefficients* of $f(\hat{\mathbf{n}})$ are

$$f_{lm} = \int d\hat{\mathbf{n}}\, f(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}) \,. \tag{1.21}$$

There are various phase conventions for the Y_{lm}; here we adopt $Y^*_{lm} = (-1)^m Y_{l\,-m}$ so that $f^*_{lm} = (-1)^m f_{l\,-m}$ for a real field.

What is the implication of statistical isotropy for the correlators of f_{lm}? For the two-point correlator, it turns out that we must have[2]

$$\langle f_{lm} f^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'} \,, \tag{1.22}$$

where C_l is the *angular power spectrum* of f. What does this imply for the two-point

[2] A plausibility argument is as follows. Under rotations, the subset of the Y_{lm} with a given l (so $2l+1$ elements) transforms irreducibly so the $\delta_{ll'}$ form of the correlator is preserved under rotation. For rotation through γ about the z-axis,

$$Y_{lm}(\theta,\phi) \to Y_{lm}(\theta, \phi - \gamma) = e^{-im\gamma} Y_{lm}(\theta,\phi) \qquad \Rightarrow \qquad f_{lm} \to e^{-im\gamma} f_{lm} \,.$$

Under rotations,

$$\langle f_{lm} f^*_{l'm'} \rangle \to e^{-im\gamma} e^{im'\gamma} \langle f_{lm} f^*_{l'm'} \rangle \,,$$

so invariance requires the correlator be $\propto \delta_{mm'}$.

correlation function? We have

$$\begin{aligned}\langle f(\hat{\mathbf{n}})f(\hat{\mathbf{n}}')\rangle &= \sum_{lm}\sum_{l'm'}\underbrace{\langle f_{lm}f^*_{l'm'}\rangle}_{C_l\delta_{ll'}\delta_{mm'}}Y_{lm}(\hat{\mathbf{n}})Y^*_{l'm'}(\hat{\mathbf{n}}')\\ &= \sum_l C_l \underbrace{\sum_m Y_{lm}(\hat{\mathbf{n}})Y^*_{lm}(\hat{\mathbf{n}}')}_{\frac{2l+1}{4\pi}P_l(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')} = C(\theta)\,, \end{aligned} \tag{1.23}$$

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \cos\theta$ and we used the addition theorem for spherical harmonics to express the sum of products of the Y_{lm} in terms of the Legendre polynomials $P_l(x)$. It follows that the two-point correlation function depends only on the angle between the two points, as required by statistical isotropy. Note that the variance of the field is

$$C(0) = \sum_l \frac{2l+1}{4\pi}C_l \approx \int d\ln l\,\frac{l(l+1)C_l}{2\pi}\,. \tag{1.24}$$

It is conventional to plot $l(l+1)C_l/(2\pi)$ which we see is the contribution to the variance per log range in l. Finally, we note that we can invert the correlation function to get the power spectrum by using orthogonality of the Legendre polynomials:

$$C_l = 2\pi\int_{-1}^{1} d\cos\theta\, C(\theta)P_l(\cos\theta)\,. \tag{1.25}$$

2 Relativistic linear theory of scalar perturbations in the conformal-Newtonian gauge

A Newtonian treatment of the growth of inhomogeneities in cosmology is inadequate on scales larger than the Hubble radius, and for relativistic fluids (like tightly-coupled radiation relevant for the CMB). The correct description requires a full general-relativistic treatment. This section provides a brief review of relativistic linear perturbation theory of scalar fluctuations in the conformal-Newtonian gauge. Mathematically, scalar perturbations are such that all perturbed quantities can be written in terms of spatial derivatives of scalar potentials. Physically, scalar perturbations describe the clumping of matter. Given the small amplitue of the fluctuations in the CMB, linear theory is adequate for most calculations in CMB physics.

We begin by recalling the background metric and dynamics. The spatially-flat FRW metric is

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j) = a^2\eta_{\mu\nu}dx^\mu dx^\nu\,. \tag{2.1}$$

Here, η is conformal time related to proper time by $dt = a d\eta$, and $a(\eta)$ is the scale factor. To avoid unnecessary complications, we shall only consider flat ($K = 0$) universes here. The Friedmann equations for such models are, in conformal time,

$$\mathcal{H}^2 = \frac{1}{3}a^2(8\pi G\rho + \Lambda) \tag{2.2}$$

$$\dot{\mathcal{H}} = \frac{1}{6}a^2[2\Lambda - 8\pi G(\rho + 3P)]\,, \tag{2.3}$$

where $\mathcal{H}$ is the conformal Hubble parameter, $\mathcal{H} \equiv \dot{a}/a = aH$ with H the proper Hubble parameter, and overdots denote differentiation with respect to conformal time.

2.1 Metric perturbations

The basic idea of relativistic perturbation theory is straightforward: perturb the metric and stress-energy tensor in the Einstein equations about their Friedmann-Robertson-Walker (FRW) forms, and, for linear perturbations, drop products of small quantities. We then solve the coupled system of equations

$$\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu} + \Lambda\delta g_{\mu\nu}\,. \tag{2.4}$$

Even in linear theory there are some technical complexities due to the *gauge freedom* inherent in general relativity (i.e. the freedom over choice of coordinates). Provided that physically-defined quantities are calculated, it does not matter which gauge is adopted – you will get the same answer for all choices. The gauge freedom can be used to our advantage by making smart choices that bring the underlying physics to the fore.

Here, we shall use the conformal-Newtonian gauge in which the spatial metric is isotropic and the 4-velocity of observers at rest in the coordinates is orthogonal to the spatial hypersurfaces of constant coordinate time. For linear scalar perturbations, this choice can always be made. Since the wordlines of coordinate observers are hypersurface orthogonal, the relative motion of these observers is free of vorticity. Moreover, since the spatial sections are isotropic, the relative motion is also shear free. The perturbed metric in the conformal-Newtonian gauge takes the simple form

$$ds^2 = a^2(\eta)\left[(1+2\psi)d\eta^2 - (1-2\phi)\delta_{ij}dx^i dx^j\right]\,. \tag{2.5}$$

For perturbations that decay at spatial infinity, the conformal Newtonian gauge is unique (i.e. the gauge is fixed). The gravitational potentials ϕ and ψ encode the two *physical* degrees of freedom in the metric of scalar perturbations. Note the similarity of the metric to the usual weak-field limit of general relativity about Minkowski space.

Orthonormal frame vectors

In calculations of the CMB anisotropies in the course, we shall make use of an *orthonormal frame* of 4-vectors, $(E_0)^\mu$ and $(E_i)^\mu$, in the perturbed metric. We take the timelike $(E_0)^\mu$ to be the 4-velocity u^μ of an observer at rest relative to the coordinate system. It follows that $(E_0)^\mu$ must be parallel to δ_0^μ and normalising gives, at linear order,

$$(E_0)^\mu = a^{-1}(1-\psi)\delta_0^\mu \,. \tag{2.6}$$

For the spatial triad, we can take

$$(E_i)^\mu = a^{-1}(1+\phi)\delta_i^\mu \,. \tag{2.7}$$

2.2 Matter perturbations

Recall that the energy and momentum of the matter is described by the stress-energy tensor. In an orthonormal frame, the components of $T^{\mu\nu}$ are

$$\begin{aligned} T^{\hat{0}\hat{0}} &= \bar{\rho}(\eta) + \delta\rho && \text{energy density} \\ T^{\hat{0}\hat{i}} &= q^i && \text{momentum density} \\ T^{\hat{i}\hat{j}} &= [\bar{P}(\eta) + \delta P]\delta^{ij} - \Pi^{ij} && \text{flux of } i\text{th component of 3-momentum} \\ & && \text{along } j\text{th direction}\,, \end{aligned} \tag{2.8}$$

where Π^{ij} is the trace-free *anisotropic stress*. We shall use the perturbations $\delta\rho \equiv \bar{\rho}(\eta)\delta$, δP, q^i and Π^{ij} to describe the perturbations to the matter and fields present, and their components are defined on the orthonormal frame in Eqs (2.6) and (2.7). However, it will be convenient to construct the coordinate components of $T^{\mu\nu}$ which are given by

$$T^{\mu\nu} = (E_\alpha)^\mu (E_\beta)^\nu T^{\hat{\alpha}\hat{\beta}} \,. \tag{2.9}$$

In the conformal-Newtonian gauge, these are

$$T^{00} = a^{-2}\bar{\rho}(1+\delta-2\psi)\,, \tag{2.10}$$

$$T^{0i} = a^{-2}q^i\,, \tag{2.11}$$

$$T^{ij} = a^{-2}\left[\bar{P}\delta^{ij} + (2\bar{P}\phi + \delta P)\delta^{ij} - \Pi^{ij}\right] \,. \tag{2.12}$$

Note how these components mix the perturbations to the stress-energy tensor with the metric perturbations. Things look neater in term of the mixed coordinate components:

$$T^0{}_0 = \bar{\rho}(1+\delta)\,, \tag{2.13}$$

$$T^i{}_0 = q^i \tag{2.14}$$

$$T^i{}_j = -(\bar{P}+\delta P)\delta^i_j + \Pi^i{}_j\,, \tag{2.15}$$

where $\Pi^i{}_j \equiv \delta_{jk}\Pi^{ik}$. Generally, we adopt the useful convention that *Latin indices on spatial vectors and tensors are raised and lowered with* δ_{ij}.

We can gain some insight into the momentum density by considering a perfect fluid with 4-velocity u^μ which is a small perturbation to the 4-velocity of observers at rest in our coordinates. The stress-energy tensor for a perfect fluid is

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu - P\delta^\mu_\nu \,. \tag{2.16}$$

Since the 4-velocity $u^\mu = dx^u/d\tau$, where τ is the proper time, and $g_{\mu\nu}u^\mu u^\nu = 1$, we have

$$\begin{aligned} 1 &= g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ &= g_{\mu\nu}\left(\frac{d\eta}{d\tau}\right)^2\frac{dx^\mu}{d\eta}\frac{dx^\nu}{d\eta} \\ &= \left(\frac{d\eta}{d\tau}\right)^2\left(g_{00} + g_{ij}\frac{dx^i}{d\eta}\frac{dx^j}{d\eta}\right) \,. \end{aligned} \tag{2.17}$$

If we write the *coordinate velocity* $dx^i/d\eta = v^i$ and assume this is a small perturbation, then Eq. (2.17) reduces to

$$1 = \left(\frac{d\eta}{d\tau}\right)^2 g_{00} = a^2(1+2\psi)\left(\frac{d\eta}{d\tau}\right)^2 \qquad \Rightarrow \qquad \frac{d\eta}{d\tau} = \frac{1}{a}(1-\psi) \tag{2.18}$$

at linear order. The components of the fluid's 4-velocity are then[3]

$$u^\mu = a^{-1}[1-\psi, v^i] \,, \tag{2.19}$$

and

$$u_0 = g_{00}u^0 + g_{0i}u^i = a^2(1+2\psi)a^{-1}(1-\psi) + O(2) = a(1+\psi) \tag{2.20}$$

$$u_i = g_{i0}u^0 + g_{ij}u^j = -a^2\delta_{ij}a^{-1}v^j = -av_i \,. \tag{2.21}$$

Using these expressions for the components u^μ and u_μ in Eq. (2.16), we find

$$q^i = T^i{}_0 = (\rho + P)a^{-1}v^i a(1+\psi) = (\rho + P)v^i \,. \tag{2.22}$$

Generally, we can define an effective peculiar velocity of the matter (for any component, or the total) by

$$q^i \equiv (\bar\rho + \bar P)v^i \,. \tag{2.23}$$

We end this section by noting that for scalar perturbations, v^i (or q^i) can be written as the gradient of a scalar: $v^i = \delta^{ij}\partial_j v$. The equivalent construction for a trace-free, symmetric 3-tensor like Π_{ij} is

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\Pi \,. \tag{2.24}$$

[3]Note that $u^\mu = (E_0)^\mu + v^i(E_i)^\mu + O(2)$ so that v^i are the orthonormal-frame components of the 3-velocity.

2.3 Field equations for scalar perturbations in the conformal Newtonian gauge

Perturbed connection coefficients

To derive the field equations, we first require the perturbed connection coefficients. Generally,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\kappa} \left(\partial_\nu g_{\kappa\rho} + \partial_\rho g_{\kappa\nu} - \partial_\kappa g_{\nu\rho} \right) . \tag{2.25}$$

The metric in the conformal-Newtonian gauge is diagonal so simple to invert:

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 - 2\psi & 0 \\ 0 & -(1+2\phi)\delta^{ij} \end{pmatrix} . \tag{2.26}$$

We therefore have

$$\Gamma^0_{00} = \mathcal{H} + \dot{\psi} , \tag{2.27}$$
$$\Gamma^0_{0i} = \partial_i \psi , \tag{2.28}$$
$$\Gamma^i_{00} = \delta^{ij} \partial_j \psi , \tag{2.29}$$
$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\dot{\phi} + 2\mathcal{H}(\phi + \psi) \right] \delta_{ij} , \tag{2.30}$$
$$\Gamma^i_{j0} = \mathcal{H}\delta^i_j - \dot{\phi}\delta^i_j , \tag{2.31}$$
$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\phi + \delta_{jk}\delta^{il}\partial_l \phi . \tag{2.32}$$

Stress-energy conservation

The field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$ imply conservation of energy and momentum via the contracted Bianchi identity:

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0 . \tag{2.33}$$

This conservation law is the relativistic version of the continuity and Euler equations in Newtonian hydrodynamics. It is more convenient to work with the mixed components $\nabla_\mu T^\mu{}_\nu = 0$ or, explicitly,

$$\partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\rho} T^\rho{}_\nu - \Gamma^\rho_{\mu\nu} T^\mu{}_\rho = 0 . \tag{2.34}$$

The 0 component, which will give the conservation of energy, is (correct to linear order) Substituting for the perturbed stress-energy tensor and the connection coefficients gives

$$\dot{\bar{\rho}} + \partial_\eta \delta\rho + \partial_i q^i + 3\mathcal{H}(\bar{\rho} + \delta\rho) - 3\bar{\rho}\dot{\phi} + 3\mathcal{H}(\bar{P} + \delta P) - 3\bar{P}\dot{\phi} = 0 . \tag{2.35}$$

The zero-order part gives the conservation of energy in the background,

$$\dot{\bar{\rho}} + 3\mathcal{H}(\bar{\rho} + \bar{P}) = 0\,, \tag{2.36}$$

and the first-order part gives

$$\partial_\eta \delta\rho + \partial_i q^i - 3(\bar{\rho} + \bar{P})\dot{\phi} + 3\mathcal{H}(\delta\rho + \delta P) = 0\,. \tag{2.37}$$

The term $\partial_i q^i$ describes changes in energy in a volume due to flow through the surfaces; the remaining two terms describe the perturbation to the dilution of energy density by expansion (including PV work) with $-3\dot{\phi}$ arising from the conformal time derivative of the perturbation to the spatial volume element $(1 - 3\phi)d^3\mathbf{x}$. If we write $\delta\rho$ in terms of the fractional overdensity, $\delta\rho = \bar{\rho}\delta$, and q^i in terms of the peculiar velocity, $q^i = (\bar{\rho} + \bar{P})v^i$, Eq. (2.37) becomes

$$\dot{\delta} + \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\partial_i v^i - 3\dot{\phi}) + 3\mathcal{H}\left(\frac{\delta P}{\bar{\rho}} - \frac{\bar{P}}{\bar{\rho}}\delta\right) = 0\,. \tag{2.38}$$

In the limit $P \ll \rho$, we recover the Newtonian continuity equation in conformal time, $\dot{\delta} + \partial_i v^i - 3\dot{\phi} = 0$, but with a general-relativistic correction due to the perturbation to the rate of expansion of space.

The ith component of Eq. (2.34) gives the conservation of 3-momentum:

$$\dot{q}_i + \partial_i \delta P - \partial_j \Pi^j{}_i + 4\mathcal{H}q_i + (\bar{\rho} + \bar{P})\partial_i\psi = 0\,. \tag{2.39}$$

Writing $q_i = (\bar{\rho}+\bar{P})v_i$, and using Eq. (2.36), we get the relativistic version of the Euler equation:

$$\dot{v}_i + \frac{1}{\bar{\rho} + \bar{P}}\partial_i \delta P - \frac{1}{\bar{\rho} + \bar{P}}\partial_j \Pi^j{}_i + \mathcal{H}v_i + \frac{\dot{\bar{P}}}{\bar{\rho} + \bar{P}}v_i + \partial_i\psi = 0\,. \tag{2.40}$$

This is like the Euler equation for a viscous fluid, with pressure gradients, the divergence of the anisotropic stress and gravitational infall driving $\dot{v}_i$, but with corrections due to redshifting of peculiar velocities ($\mathcal{H}v_i$) and P/ρ effects.

Once an equation of state of the matter (and other constitutive relations) are specified, we just need the gravitational potentials ψ and ϕ to close the system of equations of energy conservation.

Application to the tightly-coupled photon-baryon plasma

The dynamics of the pre-recombination photon-baryon plasma play a key role in CMB physics. In particular, it imprints a characteristic scale in the energy density of photons and baryons at the time of recombination – the *sound horizon*. This is simply the distance a sound wave travels in the tightly-coupled plasma since the big bang.

On comoving scales larger than a few tens of Mpc, the photons and baryons can be regarded as tightly coupled to each other through Thomson scatttering all the way until recombination. In this limit, the bulk velocities of the photons and baryons are equal and we can assign the velocity $v_{i,\gamma}$ to the total fluid. The relativistic Euler equation (2.40) does not apply to the individual photon and baryon fluids since they are exchanging momentum through scattering. However, it does apply to the total fluid which conserves its energy and momentum. Noting that $\delta P = \delta\rho_\gamma/3$ and $\dot{\bar{P}} = -4\mathcal{H}\bar{P}$ since baryon pressure can be neglected, we have

$$\dot{v}_{i,\gamma} + \frac{\mathcal{H}R}{1+R}v_{i,\gamma} + \frac{1}{4(1+R)}\partial_i\delta_\gamma + \partial_i\psi = 0\,. \tag{2.41}$$

Here, we have defined $R \equiv 3\bar{\rho}_b\,(4\bar{\rho}_\gamma)$ which is just the scale factor a normalised to 3/4 at the time of equal energy density in the baryons and photons.

It can be shown that the typical fractional energy exchange per scattering is $k_B(\bar{T}_\gamma - \bar{T}_b)/(m_e c^2)$, where $\bar{T}_\gamma$ and $\bar{T}_b$ are the photon and baryon temperatures, respectively. This it at most of order eV/MeV at recombination and so is negligible. The relativistic continuity equation (2.38) therefore applies to the photons and baryons separately giving

$$\dot{\delta}_\gamma + \frac{4}{3}\partial_i v^i_\gamma - 4\dot{\phi} = 0\,, \tag{2.42}$$

$$\dot{\delta}_b + \partial_i v^i_b - 3\dot{\phi} = 0\,. \tag{2.43}$$

Subtracting gives $\dot{\delta}_\gamma/4 = \dot{\delta}_b/3$. As we discuss in Sec. 2.6, simple inflation models predict *adiabatic* initial conditions for which $\delta_\gamma/4 = \delta_b/3$ on super-Hubble scales. We see that this condition is preserved in time on all scales for which tight-coupling holds. Physically, the adiabatic condition means that the fractional perturbations in the number densities of photons and baryons are equal (since $n_b \propto \rho_b$ but $n_\gamma \propto T^3_\gamma \propto \rho^{3/4}_\gamma$). Combining Eqs (2.41) and (2.42) gives

$$\ddot{\Theta}_0 + \frac{\mathcal{H}R}{1+R}\dot{\Theta}_0 - \frac{1}{3(1+R)}\nabla^2\Theta_0 = \ddot{\phi} + \frac{\mathcal{H}R}{1+R}\dot{\phi} + \frac{1}{3}\nabla^2\psi\,, \tag{2.44}$$

were we have defined $\Theta_0 \equiv \delta_\gamma/4$ which is the monopole of the temperature anisotropy in the photons. Equation (2.44) is a damped oscillator equation driven by gravity. The damping arises from Hubble drag on the baryons which tries to make $v_{i,b}$ decay as $1/a$. The restoring force is provided by baryon pressure and we see that the effective sound speed has $1/c_s^2 = 3(1+R)$. Note that this is reduced from the value for a photon gas by the inertia of the baryons. We shall discuss the solutions of Eq. (2.44) in the CMB course. Here, it is enough to note that in the WKB limit, the undriven equation has oscillatory solutions at a scale-dependent frequency with amplitude that varies slowly (on a Hubble time) due to the Hubble drag. Constant gravitational potentials offset the midpoint of the oscillations to $-(1+R)\psi$ which turns out to be important for the relative heights of the acoustic peaks in the angular power spectrum of the CMB anisotropies.

Perturbed Einstein equation

We require the perturbation to the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/2$, so we first need to calculate the perturbed Ricci tensor $R_{\mu\nu}$ and scalar R. The Ricci tensor is a contraction of the Riemann tensor and can be expressed in terms of the connection as

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\alpha} \,. \tag{2.45}$$

Using the perturbed connection coefficients given above, we find

$$R_{00} = -3\dot{\mathcal{H}} + \nabla^2\psi + 3\mathcal{H}(\dot{\phi} + \dot{\psi}) + 3\ddot{\phi} \,, \tag{2.46}$$

$$R_{0i} = 2\partial_i\dot{\phi} + 2\mathcal{H}\partial_i\psi \,, \tag{2.47}$$

$$R_{ij} = \left[\dot{\mathcal{H}} + 2\mathcal{H}^2 - \ddot{\phi} + \nabla^2\phi - 2(\dot{\mathcal{H}} + 2\mathcal{H}^2)(\phi + \psi) - \mathcal{H}\dot{\psi} - 5\mathcal{H}\dot{\phi}\right]\delta_{ij} + \partial_i\partial_j(\phi - \psi) \,. \tag{2.48}$$

We now form the Ricci tensor $g^{\mu\nu}R_{\mu\nu}$ using

$$R = g^{00}R_{00} + 2\underbrace{g^{0i}R_{0i}}_{0} + g^{ij}R_{ij} \,. \tag{2.49}$$

It follows that, to linear order,

$$a^2R = -6(\dot{\mathcal{H}} + \mathcal{H}^2) + 2\nabla^2\psi - 4\nabla^2\phi + 12(\dot{\mathcal{H}} + \mathcal{H}^2)\psi + 6\ddot{\phi} + 6\mathcal{H}(\dot{\psi} + 3\dot{\phi}) \,. \tag{2.50}$$

Finally, we can form the Einstein tensor. The 00 component

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\phi - 6\mathcal{H}\dot{\phi} \,. \tag{2.51}$$

The $0i$ component of the Einstein tensor is simply R_{0i} since $g_{0i} = 0$ in the conformal-Newtonian gauge:

$$G_{0i} = 2\partial_i\dot{\phi} + 2\mathcal{H}\partial_i\psi \,. \tag{2.52}$$

The remaining components are

$$G_{ij} = -(2\dot{\mathcal{H}} + \mathcal{H}^2)\delta_{ij} + \left[\nabla^2(\psi - \phi) + 2\ddot{\phi} + 2(2\dot{\mathcal{H}} + \mathcal{H}^2)(\phi + \psi) + 2\mathcal{H}\dot{\psi} + 4\mathcal{H}\dot{\phi}\right]\delta_{ij} + \partial_i\partial_j(\phi - \psi) \,. \tag{2.53}$$

Substituting the perturbed Einstein tensor, metric and stress-energy tensor into the Einstein equation gives the equations of motion for the metric perturbations and the zero-order Friedmann equations. For example, the zero-order part of the 00 equation gives

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar{\rho} + \frac{1}{3}\Lambda a^2 , \tag{2.54}$$

which is just the Friedmann equation (2.2). The first-order part gives

$$\nabla^2 \phi = (8\pi G a^2 \bar{\rho} + \Lambda a^2)\psi + 3\mathcal{H}\dot{\phi} + 4\pi G a^2 \bar{\rho}\delta . \tag{2.55}$$

which, on using Eq. (2.54), reduces to

$$\nabla^2 \phi = 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi) + 4\pi G a^2 \bar{\rho}\delta . \tag{2.56}$$

The $0i$ Einstein equation is identically satisfied at zero order by isotropy. The first-order part is

$$\partial_i \dot{\phi} + \mathcal{H}\partial_i \psi = -4\pi G a^2 q_i . \tag{2.57}$$

If we write $q_i = (\bar{\rho} + \bar{P})\partial_i v$ and assume the perturbations decay at infinity, we can integrate Eq. (2.57) to get

$$\dot{\phi} + \mathcal{H}\psi = -4\pi G a^2 (\bar{\rho} + \bar{P}) v . \tag{2.58}$$

Substituting this in the 00 Einstein equation gives

$$\nabla^2 \phi = 4\pi G a^2 \left[\bar{\rho}\delta - 3\mathcal{H}(\bar{\rho} + \bar{P})v\right] . \tag{2.59}$$

This is of the form of a Poisson equation but with source density $\bar{\rho}\Delta \equiv \bar{\rho}\delta - 3\mathcal{H}(\bar{\rho}+\bar{P})v$. What is the physical meaning of this term? Let us introduce *comoving hypersurfaces* as those that are orthogonal to the worldlines of a set of observers comoving with the total matter (i.e. they see $q^i = 0$) and are the constant-time hypersurfaces in the *comoving orthogonal gauge* for which $q^i = 0$ and $g_{i0} = 0$. It can be shown that Δ is the fractional overdensity in the comoving gauge and we see from Eq. (2.59) that this is the source term for the gravitational potential ϕ.

The final content of the Einstein equation is contained in its ij components. Using Eq. (2.53) in the left-hand side, the zero-order part is

$$2\dot{\mathcal{H}} + \mathcal{H}^2 = -a^2(8\pi G\bar{P} - \Lambda) , \tag{2.60}$$

which, with Eq. (2.2) recover the second Friedmann equation (2.3). The first-order part can be broken up into its trace and trace-free parts. The latter is

$$\partial_{\langle i}\partial_{j\rangle}(\phi - \psi) = -8\pi G a^2 \Pi_{ij} . \tag{2.61}$$

This shows that, *in the absence of anisotropic stress* (and assuming appropriate decay at infinity), $\phi = \psi$ so there is then only one physical degree of freedom in the metric.

For the trace part, contracting δG_{ij} with δ^{ij} and using Eq. (2.60) in the right-hand side gives

$$\ddot{\phi} + \frac{1}{3}\nabla^2(\psi - \phi) + (2\dot{\mathcal{H}} + \mathcal{H}^2)\psi + \mathcal{H}\dot{\psi} + 2\mathcal{H}\dot{\phi} = 4\pi G a^2 \delta P \,. \tag{2.62}$$

Of course, the Einstein equations and the energy and momentum conservation equations form a redundant (but consistent!) set because of the Bianchi identity. We can use whichever subsets are most convenient for the particular problem at hand.

2.4 Density perturbations of a radiation fluid

As an example of using the perturbed conservation and Einstein equations, we consider the evolution of density perturbations in a universe containing only a radiation fluid, i.e. an ideal fluid with $P = \rho/3$. This is a reasonable model of what happens in a realistic multi-component universe when it is radiation dominated, provided that the *fluctuations* in the radiation also dominate all other species[4].

The basic idea is to get a closed equation for the potential ϕ (noting that $\phi = \psi$ since $\Pi_{ij} = 0$ by assumption) and then use this to determine the density fluctuations in the radiation. We start from Eq. (2.62) with $\delta P = \delta\rho/3$. In a universe dominated by radiation, $\mathcal{H}^2 \propto a^{-2}$ hence $a \propto \eta$ and $\mathcal{H} = 1/\eta$; it follows that

$$\ddot{\phi} + \frac{3}{\eta}\dot{\phi} - \frac{1}{\eta^2}\phi = \frac{4\pi G a^2 \bar{\rho}}{3}\delta_r \,. \tag{2.63}$$

We can eliminate δ_r using the 00 Einstein equation in the form of Eq. (2.56) to find

$$\begin{aligned} &\ddot{\phi} + \frac{3}{\eta}\dot{\phi} - \frac{1}{\eta^2}\phi = \frac{1}{3}\nabla^2\phi - \frac{1}{\eta}\left(\dot{\phi} + \frac{1}{\eta}\phi\right) \\ \Rightarrow \quad &\ddot{\phi} + \frac{4}{\eta}\dot{\phi} - \frac{1}{3}\nabla^2\phi = 0 \,. \end{aligned} \tag{2.64}$$

This is a damped wave equation with propagation speed $1/\sqrt{3}$ as expected for a radiation fluid where the square of the sound speed is $\partial P/\partial\rho = 1/3$. On Fourier expanding ($\nabla^2\phi \to -k^2\phi$), Eq. (2.64) gives

$$\ddot{\phi} + \frac{4}{\eta}\dot{\phi} + \frac{k^2}{3}\phi = 0 \,. \tag{2.65}$$

[4] This is the case for adiabatic initial conditions – see Sec. 2.6 – but not for *isocurvature* initial conditions, where the radiation is initially smoothly distributed but another component, sub-dominant in the background, is not.

Finally, writing $\phi = u(x)/x$ where $x \equiv k\eta/\sqrt{3}$, we have

$$u'' + \frac{2}{x}u' + \left(1 - \frac{2}{x^2}\right) u = 0\,. \tag{2.66}$$

This has independent solutions which are just the spherical Bessel functions $j_1(x)$ and $n_1(x)$[5]. These can be written explicitly as

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} = \frac{x}{3} + O(x^3) \\ n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} = -\frac{1}{x^2} + O(x^0)\,. \end{aligned} \tag{2.67}$$

The regular (growing-mode) solution for ϕ is thus

$$\phi \propto \frac{j_1(k\eta/\sqrt{3})}{k\eta}\,, \tag{2.68}$$

and is constant outside the *sound horizon*, i.e. for $k^{-1} \gg \eta/\sqrt{3}$. The asymptotic form of the spherical Bessel functions are

$$j_l(x) \sim \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)\,, \tag{2.69}$$

hence $\phi \sim \cos(k\eta/\sqrt{3})/(k\eta)^2$ well inside the sound horizon. As expected from Eq. (2.65), we have oscillations at frequency $k/\sqrt{3}$ with a slow damping of the amplitude (over the order of a Hubble time).

The Poisson equation (2.59) relates the potential to the density perturbation in the comoving gauge. Here, this is the gauge that is comoving with the radiation fluid. Fourier expanding, we have

$$-k^2\phi = \frac{3}{2}\mathcal{H}^2\Delta_r = \frac{3}{2\eta^2}\Delta_r\,, \tag{2.70}$$

so that

$$\Delta_r = -\frac{2}{3}(k\eta)^2\phi \propto k\eta j_1(k\eta/\sqrt{3})\,. \tag{2.71}$$

This grows like $\eta^2 \propto a^2$ outside the sound horizon and oscillates with constant amplitude inside the sound horizon. These sub-horizon oscillations are what give rise to the acoustic peaks in the power spectrum of the CMB anisotropies.

[5]Generally, spherical Bessel functions j_l and n_l satisfy

$$y_l'' + \frac{2}{x}y_l' + \left(1 - \frac{l(l+1)}{x^2}\right) y_l = 0\,,$$

where y_l is j_l or n_l.

We have to work a little harder to find the density perturbation in the conformal Newtonian gauge. Using Eq. (2.56), we have

$$-k^2\phi - \frac{3}{\eta}\left(\dot{\phi} + \frac{1}{\eta}\phi\right) = \frac{3}{2\eta^2}\delta_r$$
$$\Rightarrow \quad \delta_r = -\frac{2}{3}(k\eta)^2\phi - 2\eta\dot{\phi} - 2\phi\,. \tag{2.72}$$

On large scales, $k\eta \ll 1$, ϕ is of the form $A + B(k\eta)^2$ so $\eta\dot{\phi} = 2B(k\eta)^2 \ll \phi$ and hence

$$\delta_r \approx -2\phi \quad (k\eta \ll 1) \tag{2.73}$$

and is constant. On small scales, $k\eta \gg 1$, ϕ oscillates and so $\dot{\phi} \sim k\phi$ and we find

$$\delta_r \approx -\frac{2}{3}(k\eta)^2\phi = \Delta_r\,. \tag{2.74}$$

We see that well inside the horizon, the density perturbations in the comoving and Newtonian gauge coincide. This is indicative of the general result that there are no gauge ambiguities inside the horizon.

Finally, from Eq. (2.57) we see that

$$v_i = -\frac{\eta^2}{2}\frac{\partial}{\partial x^i}\left(\dot{\phi} + \frac{1}{\eta}\phi\right), \tag{2.75}$$

so the Newtonian-gauge peculiar velocity of the radiation fluid vanishes like $-k\eta\phi/2$ on large scales.

2.5 Evolution of matter fluctuations in the matter and acceleration eras

As another important example, we consider the evolution of fluctuations in pressure-free matter in a universe dominated by such a component plus a smoothly distributed dark energy in the form of a cosmological constant. This is a good model of the universe well after matter-radiation equality. Although the density perturbations in the CDM and baryons are, generally, quite different at the time of recombination, the baryons quickly fall into the potential wells of the CDM and both components then have the same fractional density contrasts and velocities and we can consider them together as a single pressure-free fluid.

In the matter era, well after recombination, the comoving frame moves with the matter (CDM and baryons). The total density perturbation on comoving hypersurfaces is then essentially Δ_m and this is related to ϕ via

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho}_m \Delta_m\,. \tag{2.76}$$

The pressure fluctuation is negligible and so, from Eq. (2.62), the potential evolves as

$$\ddot{\phi} + 3\mathcal{H}\dot{\phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\phi = 0 \,. \tag{2.77}$$

In matter domination, $a \propto \eta^2$ and $\mathcal{H} = 2/\eta$ so $2\dot{\mathcal{H}} + \mathcal{H}^2 = 0$ and we find

$$\ddot{\phi} + \frac{6}{\eta}\dot{\phi} = 0 \,. \tag{2.78}$$

The solutions of this are $\phi =$ const. and a decaying solution $\phi \propto \eta^{-5} \propto a^{-5/2}$, as in Newtonian theory but now valid on all scales. The comoving density contrast therefore evolves as

$$\Delta_m \propto a^{-2} \underbrace{a^3}_{\text{from } \bar{\rho}} \underbrace{\begin{pmatrix} 1 \\ a^{-5/2} \end{pmatrix}}_{\text{from } \phi} \propto \begin{pmatrix} a \\ a^{-3/2} \end{pmatrix} \,. \tag{2.79}$$

These also agree with the Newtonian treatment, but note that this is for the comoving density contrast and the Newtonian-gauge result differs (the growing mode is constant) outside the horizon.

More generally, the late-time evolution of Δ_m follows from combining Eqs (2.76) and (2.77). Since $a^2\bar{\rho}_m \propto a^{-1}$, we have

$$\partial_\eta^2(\Delta_m/a) + 3\mathcal{H}\partial_\eta(\Delta/a) + (2\dot{\mathcal{H}} + \mathcal{H}^2)(\Delta_m/a) = 0 \,, \tag{2.80}$$

which rearranges to

$$\ddot{\Delta}_m + \mathcal{H}\dot{\Delta}_m + (\dot{\mathcal{H}} - \mathcal{H}^2)\Delta_m = 0 \,. \tag{2.81}$$

The Friedmann equations (2.2) and (2.3) for matter and Λ give

$$\begin{aligned} \dot{\mathcal{H}} - \mathcal{H}^2 &= \frac{1}{3}a^2\Lambda - \frac{4\pi G a^2}{3}\bar{\rho}_m - \frac{1}{3}a^2\Lambda - \frac{8\pi G a^2}{3}\bar{\rho}_m \\ &= -4\pi G a^2 \bar{\rho}_m \,, \end{aligned} \tag{2.82}$$

so, generally, the comoving density contrast evolves as

$$\ddot{\Delta}_m + \mathcal{H}\dot{\Delta}_m - 4\pi G a^2 \bar{\rho}_m \Delta_m = 0 \,. \tag{2.83}$$

Once the universe becomes dominated by dark energy, $\mathcal{H}^2 \gg 4\pi G a^2 \bar{\rho}_m$ and the third term on the left of Eq. (2.83) is small compared to the other two. It follows that there is a solution with $\Delta_m =$ const. and a decaying solution with $\dot{\Delta}_m \propto 1/a$. We see that the growth of structure halts once the universe becomes Λ dominated and, from the Poisson equation, ϕ decays as $1/a$. This decay of the potential induces the *late-time integrated Sachs-Wolfe effect* in the CMB.

2.6 Comoving curvature perturbation

There is an important quantity that is conserved on super-Hubble scales for adiabatic (defined shortly), scalar fluctuations irrespective of the equation of state of the matter: the *comoving curvature perturbation*. This is the perturbation to the intrinsic curvature scalar of comoving hypersurfaces, i.e. those hypersurfaces orthogonal to the worldlines that comove with the total matter (i.e. for which $q^i = 0$). Its importance is that it allows us to match the perturbations from inflation to those in the radiation-dominated universe on large scales without needing to know the (uncertain) details of the reheating phase at the end of inflation.

In some arbitrary gauge, let us work out the *intrinsic curvature* of surfaces of constant time. The *induced metric*, γ_{ij}, on these surfaces is generally

$$\gamma_{ij} \equiv a^2 \left[(1-2\phi)\delta_{ij} + 2E_{ij}\right] , \tag{2.84}$$

where E_{ij} is symmetric and trace-free. This metric is not Euclidean because of the perturbations ϕ and E_{ij}. The 3D metric has an associated (metric-compatible) connection

$${}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right) , \tag{2.85}$$

where γ^{ij} is the inverse of the induced metric,

$$\gamma^{ij} = a^{-2}\left[(1+2\phi)\delta^{ij} - 2E^{ij}\right] . \tag{2.86}$$

We actually only need the inverse metric to zero-order to compute the connection to first-order since the spatial derivatives of the γ_{ij} are all first-order in the perturbations. We have

$${}^{(3)}\Gamma^i_{jk} = -\left(2\delta^i_{(j}\partial_{k)}\phi - \delta^{il}\delta_{jk}\partial_l\phi\right) + \left(2\partial_{(j}E_{k)}{}^i - \delta^{il}\partial_l E_{jk}\right) . \tag{2.87}$$

We can form the intrinsic curvature tensor in the same way as in 4D spacetime. The intrinsic curvature is the associated Ricci scalar, given by

$${}^{(3)}R = \gamma^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \gamma^{ik}\partial_k {}^{(3)}\Gamma^l_{il} + \gamma^{ik(3)}\Gamma^l_{ik}{}^{(3)}\Gamma^m_{lm} - \gamma^{ik(3)}\Gamma^m_{il}{}^{(3)}\Gamma^l_{km} . \tag{2.88}$$

To first-order, we therefore have

$$a^{2(3)}R = \delta^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \delta^{ik}\partial_k {}^{(3)}\Gamma^l_{il} \tag{2.89}$$

$$= 4\nabla^2\phi + 2\partial_i\partial_j E^{ij} . \tag{2.90}$$

Note that this vanishes for vector and tensor perturbations (as do all perturbed scalars) since then $\phi = 0$ and $\partial_i\partial_j E^{ij} = 0$. For scalar perturbations, $E_{ij} = \partial_{\langle i}\partial_{j\rangle}E$ so

$$\begin{aligned}\partial_i\partial_j E^{ij} &= \delta^{il}\delta^{jm}\partial_i\partial_j\left(\partial_l\partial_m E - \frac{1}{3}\delta_{lm}\nabla^2 E\right) \\ &= \nabla^2\nabla^2 E - \frac{1}{3}\nabla^2\nabla^2 E = \frac{2}{3}\nabla^4 E .\end{aligned} \tag{2.91}$$

Finally, we find

$$a^{2(3)}R = 4\nabla^2\left(\phi + \frac{1}{3}\nabla^2 E\right). \tag{2.92}$$

We define the *curvature perturbation* as $-(\phi + \nabla^2 E/3)$ and the *comoving curvature perturbation* $\mathcal{R}$ is this quantity evaluated in the comoving gauge ($g_{i0} = 0 = q^i$). It can be shown that $\mathcal{R}$, when expressed in terms of the metric perturbations in the conformal-Newtonian gauge, is $\mathcal{R} = -\phi + \mathcal{H}v$, where the velocity v arises from the transformation between the comoving and conformal-Newtonian gauges. We can now use the $0i$ Einstein equation (2.58) to eliminate the peculiar velocity in favour of the potentials:

$$\mathcal{R} = -\phi - \frac{\mathcal{H}(\dot{\phi} + \mathcal{H}\psi)}{4\pi G a^2(\bar{\rho} + \bar{P})}. \tag{2.93}$$

Differentiating Eq. (2.93), and using the perturbed Einstein equations, gives

$$-4\pi G a^2(\bar{\rho} + \bar{P})\dot{\mathcal{R}} = \frac{1}{3}\mathcal{H}\nabla^2(\phi - \psi) + 4\pi G a^2 \mathcal{H}\delta P_{\mathrm{nad}} + \frac{\mathcal{H}\dot{\bar{P}}}{\dot{\bar{\rho}}}\nabla^2\phi. \tag{2.94}$$

Here, we have introduced the *non-adiabatic pressure perturbation*

$$\delta P_{\mathrm{nad}} \equiv \delta P - \frac{\dot{\bar{P}}}{\dot{\bar{\rho}}}\bar{\rho}\delta. \tag{2.95}$$

It is gauge-invariant in that you get the same quantity if you evaluate the right-hand side in any gauge. It vanishes for a barotropic equation of state, $P = P(\rho)$ such as in a fluid with no entropy perturbations. More generally, it vanishes for *adiabatic fluctuations* in a mixture of barotropic fluids. Adiabatic fluctuations are such that

$$\frac{\bar{\rho}_i\delta_i}{\dot{\bar{\rho}}_i} - \frac{\bar{\rho}_j\delta_j}{\dot{\bar{\rho}}_j} = 0 \tag{2.96}$$

for all species i and j. Equivalently, the density is uniform over the same spatial hypersurfaces for all species and so, for mixtures of barotropic fluids, the pressures are also all uniform over the same surfaces and so $\delta P_{\mathrm{nad}} = 0$. The adiabatic condition is preserved in time for super-Hubble scale fluctuations if the relative velocities of the fluids initially vanish – essentially a consequence of the equivalence principle. Such adiabatic initial conditions arise naturally in models of inflation with a single scalar field.

If the non-adiabatic pressure vanishes, the right-hand side of Eq. (2.94) is $\sim \mathcal{H}k^2\phi \sim \mathcal{H}k^2\mathcal{R}$ at scale k (where we have used equation 2.93), so that $\dot{\mathcal{R}}/\mathcal{H} \sim (k/\mathcal{H})^2\mathcal{R}$ and vanishes on super-Hubble scales[6].

[6] Note that the first term on the right-hand side of Eq. (2.94) vanishes on all scales if the anisotropic stress vanishes.

We saw earlier that for adiabatic fluctuations the potential is constant during radiation domination on scales larger than the sound horizon, and on all (linear) scales during matter domination. How does the potential change during the matter-radiation transition on large scales? During radiation domination, a constant potential is related to the (super-Hubble) $\mathcal{R}$ by

$$\mathcal{R} = -\phi - \frac{\mathcal{H}^2\phi}{16\pi G\bar{\rho}a^2/3} = -\phi - \frac{\mathcal{H}^2\phi}{2\mathcal{H}^2} = -\frac{3}{2}\phi \,. \tag{2.97}$$

During matter domination, we have instead

$$\mathcal{R} = -\phi - \frac{\mathcal{H}^2\phi}{4\pi G\bar{\rho}a^2} = -\phi - \frac{\mathcal{H}^2\phi}{3\mathcal{H}^2/2} = -\frac{5}{3}\phi \,. \tag{2.98}$$

Since $\mathcal{R}$ is constant, we have

$$\frac{5}{3}\phi|_{\mathrm{matter}} = \frac{3}{2}\phi|_{\mathrm{radiation}} \quad \Rightarrow \quad \phi|_{\mathrm{matter}} = \frac{9}{10}\phi|_{\mathrm{radiation}} = -\frac{3}{5}\mathcal{R} \,. \tag{2.99}$$

This result is useful for relating the amplitude of the large-angle fluctuations in the CMB to the amplitude of primordial fluctuations. The fact that ϕ decays by 10% across the radiation-matter transition, and that this decay is still not complete by recombination, leads to the *early-time integrated Sachs-Wolfe effect* in the CMB which is critically important for the height of the first acoustic peak.

In courses on inflation, you may have seen how to compute the statistics of $\mathcal{R}$ at the end of inflation in Fourier space; the main results will be summarised in this course. The statistics of $\mathcal{R}$ are related to those of any (linear) observable, since, in linear perturbation theory, different scales decouple so the Fourier transform of some observable, such as $\delta(\mathbf{k})$, will be linear in the primordial $\mathcal{R}(\mathbf{k})$.

3 Boltzmann equation for Thomson scattering of unpolarized radiation

In the rest frame of the electrons, the scattering rate (in terms of proper time $\tilde{\tau}$ in that frame) is

$$\left.\frac{d\tilde{f}(\tilde{\epsilon},\tilde{\mathbf{e}})}{d\tilde{\tau}}\right|_{\mathrm{scatt.}} = -n_{\mathrm{e}}\sigma_{\mathrm{T}}\tilde{f}(\tilde{\epsilon},\tilde{\mathbf{e}}) + \frac{3n_{\mathrm{e}}\sigma_{\mathrm{T}}}{16\pi}\int d\tilde{\mathbf{e}}_{\mathrm{in}}\,\tilde{f}(\tilde{\epsilon},\tilde{\mathbf{e}}_{\mathrm{in}})\left[1+(\tilde{\mathbf{e}}_{\mathrm{in}}\cdot\tilde{\mathbf{e}})^2\right] \,. \tag{3.1}$$

Here, tildes denote quantities in the electrons' rest frame and we have used the fact that there is no energy transfer in Thomson scattering in the rest frame. The photon directions in the rest-frame are expressed in terms of components on the Lorentz-boosted CNG orthonormal frame. To zero-order in perturbations, the distribution function is isotropic in the rest frame and the out-scattering rate (the first term on the right of Eq. 3.1) balances the in-scattering rate (the second term on the right). The scattering rate is therefore first-order in perturbations and we only need the electron density n_e at zero-order. Moreover, we can neglect the difference in proper time between the rest frame and the CNG frame in transforming the scattering rate back to the CNG frame since this first-order difference multiplies the first-order scattering rate. With these observations, and using the Lorentz invariance of the distribution function, we can write the scattering rate with respect to conformal time in the CNG frame as

$$\left.\frac{df(\epsilon,\mathbf{e})}{d\eta}\right|_{\mathrm{scatt.}} = -a\bar{n}_\mathrm{e}\sigma_\mathrm{T} f(\epsilon,\mathbf{e}) + \frac{3a\bar{n}_\mathrm{e}\sigma_\mathrm{T}}{16\pi}\int d\tilde{\mathbf{e}}_\mathrm{in}\,\tilde{f}(\tilde{\epsilon},\tilde{\mathbf{e}}_\mathrm{in})\left[1+(\tilde{\mathbf{e}}_\mathrm{in}\cdot\tilde{\mathbf{e}})^2\right]. \tag{3.2}$$

The in-scattering process on the right of Eq. (3.2) involves the following direction and energy changes in the two frames:

$$\begin{aligned} &\text{CNG:} && (\epsilon_\mathrm{in},\mathbf{e}_\mathrm{in}) \to (\epsilon,\mathbf{e}) \\ &\text{Rest frame:} && (\tilde{\epsilon}_\mathrm{in},\tilde{\mathbf{e}}_\mathrm{in}) \to (\tilde{\epsilon},\tilde{\mathbf{e}}) \quad \text{with } \tilde{\epsilon}_\mathrm{in} = \tilde{\epsilon}\,. \end{aligned} \tag{3.3}$$

Applying standard Lorentz transformations gives

$$\tilde{\epsilon} = \gamma\epsilon(1-\mathbf{e}\cdot\mathbf{v}_b)\,, \tag{3.4}$$

and, using the inverse transformation,

$$\begin{aligned} \epsilon_\mathrm{in} &= \gamma\tilde{\epsilon}_\mathrm{in}(1+\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v}) \\ &= \gamma^2\epsilon(1+\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v})(1-\mathbf{e}\cdot\mathbf{v}_b) \\ &\approx \epsilon(1+\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v}-\mathbf{e}\cdot\mathbf{v}_b)\,, \end{aligned} \tag{3.5}$$

where γ is the Lorentz factor between the two frames. We now use the Loretntz invariance of f to write (to first order)

$$\begin{aligned} \tilde{f}(\tilde{\epsilon},\tilde{\mathbf{e}}_\mathrm{in}) &= f(\epsilon_\mathrm{in},\mathbf{e}_\mathrm{in}) \\ &= f[\epsilon(1+\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v}_b-\mathbf{e}\cdot\mathbf{v}_b),\mathbf{e}_\mathrm{in}] \\ &= \bar{f}[\epsilon(1+\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v}_b-\mathbf{e}\cdot\mathbf{v}_b)] - \frac{d\bar{f}}{d\ln\epsilon}\Theta(\mathbf{e}_\mathrm{in}) \\ &= \bar{f}(\epsilon) + \frac{d\bar{f}}{d\ln\epsilon}(\tilde{\mathbf{e}}_\mathrm{in}-\mathbf{e})\cdot\mathbf{v}_b - \frac{d\bar{f}}{d\ln\epsilon}\Theta(\mathbf{e}_\mathrm{in})\,. \end{aligned} \tag{3.6}$$

Substituting into the in-scattering part of Eq. (3.2), the $\bar{f}(\epsilon)$ term integrates to give $a\bar{n}_\mathrm{e}\sigma_\mathrm{T}\bar{f}(\epsilon)$. The third term on the right of Eq. (3.6) is first-order and so we can replace $d\tilde{\mathbf{e}}_\mathrm{in}$ with $d\mathbf{e}_\mathrm{in}$ and $\tilde{\mathbf{e}}_\mathrm{in}\cdot\tilde{\mathbf{e}}$ with $\mathbf{e}_\mathrm{in}\cdot\mathbf{e}$ in the angular integral. Finally, for the remaining term in Eq. (3.6) the $\tilde{\mathbf{e}}_\mathrm{in}\cdot\mathbf{v}_b$ part integrates to zero by (parity) symmetry while the

$-\mathbf{e}\cdot\mathbf{v}_b$ part is independent of $\tilde{\mathbf{e}}_{\mathrm{in}}$ and can be taken outside the angular integral. Putting these pieces together, we find

$$\left.\frac{df(\epsilon,\mathbf{e})}{d\eta}\right|_{\mathrm{scatt.}} = \frac{d\bar{f}}{d\ln\epsilon}\left(a\bar{n}_{\mathrm{e}}\sigma_{\mathrm{T}}\Theta(\mathbf{e}) - \frac{3a\bar{n}_{\mathrm{e}}\sigma_{\mathrm{T}}}{16\pi}\int d\hat{\mathbf{m}}\Theta(\hat{\mathbf{m}})\left[1+(\mathbf{e}\cdot\hat{\mathbf{m}})^2\right] - a\bar{n}_{\mathrm{e}}\sigma_{\mathrm{T}}\mathbf{e}\cdot\mathbf{v}_b\right) . \tag{3.7}$$

This is our required expression for the scattering rate in the presence of non-zero electron bulk velocity. Note that in-scattering into the direction of the electron velocity *increases* the temperature anisotropy along that direction.

4 CMB anisotropies from gravitational waves

Tensor modes, describing gravitational waves, represent the transverse trace-free perturbations to the spatial metric:

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij}+h_{ij})dx^i dx^j] , \tag{4.1}$$

with $h^i_i = 0$ and $\partial_i h^i_j = 0$. A convenient parameterisation of the photon four-momentum in this case is

$$p^\mu = \frac{\epsilon}{a^2}\left[1, e^i - \frac{1}{2}h^i_j e^j\right] , \tag{4.2}$$

where, as for scalar perturbations, $\mathbf{e}^2 = 1$ and ϵ is a times the energy of the photon as seen by an observer at constant $\mathbf{x}$. The components of $\mathbf{e}$ are the direction cosines of the photon direction for this observer on an orthonormal spatial triad of vectors $a^{-1}(\partial_i - h^j_i\partial_j/2)$, and are constant in the unperturbed universe. The evolution of the comoving energy, ϵ, follows from the geodesic equation:

$$\frac{1}{\epsilon}\frac{d\epsilon}{d\eta} + \frac{1}{2}\dot{h}_{ij}e^i e^j = 0 , \tag{4.3}$$

and so the Boltzmann equation for the fractional temperature anisotropies is

$$\begin{aligned}\frac{\partial\Theta}{\partial\eta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta &= -an_{\mathrm{e}}\sigma_{\mathrm{T}}\Theta + \frac{3an_{\mathrm{e}}\sigma_{\mathrm{T}}}{16\pi}\int d\hat{\mathbf{m}}\,\Theta(\hat{\mathbf{m}})\left[1+(\mathbf{e}\cdot\hat{\mathbf{m}})^2\right] \\ &\quad - \frac{1}{2}\dot{h}_{ij}e^i e^j .\end{aligned} \tag{4.4}$$

All perturbed scalar and vector quantities vanish for tensor perturbations so $\Theta(\mathbf{e})$ has only $l \geq 2$ moments. Neglecting the anisotropic nature of Thomson scattering, or, equivalently, the temperature quadrupole at last scattering, the solution of Eq. (4.4) is an integral along the unperturbed line of sight:

$$[\Theta(\mathbf{e})]_R = -\frac{1}{2}\int_{\eta_*}^{\eta_R} e^{-\tau}\dot{h}_{ij}e^i e^j\,d\eta . \tag{4.5}$$

The physics behind this solution is as follows. The time derivative $\dot{h}_{ij}$ is the shear induced by the gravitational waves. This quadrupole perturbation to the expansion rate produces an anisotropic redshifting of the CMB photons and an associated temperature anisotropy.

The shear source term for the tensor anisotropies is locally a quadrupole since $\dot{h}_{ij}$ is trace-free. Projecting this contribution at a given time onto angular multipoles on the sky is a little more involved than for scalar perturbations. We start by decomposing $\dot{h}_{ij}$ into circularly-polarized Fourier modes,

$$h_{ij} = \sum_{\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} h_{ij}^{(\pm 2)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \qquad h_{ij}^{(\pm 2)}(\mathbf{k}) = \frac{1}{\sqrt{2}} m_{ij}^{(\pm 2)}(\mathbf{k}) h^{(\pm 2)}(\mathbf{k}) \, . \tag{4.6}$$

Here, the basis tensors $m_{ij}^{(\pm 2)}(\mathbf{k})$ are complex, symmetric trace-free and orthogonal to $\mathbf{k}$, and $h^{(\pm 2)}(\mathbf{k})$ are scalar Fourier amplitudes. For $\mathbf{k}$ along the z-axis, we choose $m_{ij}^{(\pm 2)} = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})_i (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})_j/2$. For such Fourier modes the shear source is

$$\dot{h}_{ij}^{(\pm 2)}(k\hat{\mathbf{z}}) e^i e^j \propto \frac{1}{2\sqrt{2}} \left[(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \cdot \mathbf{e}\right]^2 = \frac{1}{2\sqrt{2}} \sin^2\theta e^{\pm 2i\phi} = \sqrt{\frac{4\pi}{15}} Y_{2\,\pm 2}(\mathbf{e}) \, , \tag{4.7}$$

where θ and ϕ are the polar coordinates associated with the direction $\mathbf{e}$. The projection of the shear source at comoving distance χ is

$$\begin{aligned} \sqrt{\frac{4\pi}{15}} Y_{2\,\pm 2}(\mathbf{e}) e^{-ik\chi\cos\theta} &= \frac{4\pi}{\sqrt{15}} \sum_L (-i)^L \sqrt{2L+1} j_L(k\chi) Y_{2\,\pm 2}(\mathbf{e}) Y_{L0}(\mathbf{e}) \\ &= \sqrt{\frac{4\pi}{3}} \sum_L \Bigg[(-i)^L (2L+1) j_L(k\chi) \\ &\times \sum_l \sqrt{2l+1} \begin{pmatrix} 2 & L & l \\ \mp 2 & 0 & \pm 2 \end{pmatrix} \begin{pmatrix} 2 & L & l \\ 0 & 0 & 0 \end{pmatrix} Y_{l \pm 2}(\mathbf{e}) \Bigg] \, , \end{aligned} \tag{4.8}$$

where we have used the Rayleigh plane-wave expansion. The $3j$ symbols arise from coupling the angular dependence of the source to that of the plane wave. Since they force L and l to have the same parity, the lth multipoles of the projection involve the l and $l \pm 2$ multipoles of the plane wave. We can simplify further by writing out the $3j$ symbols explicitly and using the recursion relations for Bessel functions to express $j_{l\pm 2}$ in terms of j_l. The final result is remarkably compact:

$$\dot{h}_{ij}^{(\pm 2)}(k\hat{\mathbf{z}}) e^i e^j e^{-ik\chi\cos\theta} = -\sqrt{\frac{\pi}{2}} \dot{h}^{(\pm 2)}(k\hat{\mathbf{z}}) \sum_l (-i)^l \sqrt{2l+1} \sqrt{\frac{(l+2)!}{(l-2)!}} \frac{j_l(k\chi)}{(k\chi)^2} Y_{l\,\pm 2}(\mathbf{e}) \, . \tag{4.9}$$

The spatial-to-angular projection is now controlled by $j_l(k\chi)/(k\chi)^2$. As for scalar perturbations, this is concentrated on multipoles $l \sim k\chi$. The result for general $\mathbf{k}$ can be obtained by a rotation $\hat{D}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, 0)$, where $\theta_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ are the polar and azimuthal angles of $\hat{\mathbf{k}}$ and the third Euler angle can be taken to be zero. Pre-empting a little the discussion of polarization in the second half of this course, we can relate the rotation matrices to spin-weighted spherical harmonics, $D^l_{m\,\pm 2}(\phi_{\mathbf{k}}, \theta_{\mathbf{k}}, 0) = \sqrt{4\pi/(2l+1)}\,{}_{\mp 2}Y^*_{lm}(\hat{\mathbf{k}})$, so that

$$\dot{h}^{(\pm 2)}_{ij}(\mathbf{k}) e^i e^j e^{-i\chi \mathbf{k}\cdot\mathbf{e}} = -\sqrt{2\pi^2}\dot{h}^{(\pm 2)}(\mathbf{k}) \sum_{lm} (-i)^l \sqrt{\frac{(l+2)!}{(l-2)!}} \frac{j_l(k\chi)}{(k\chi)^2} {}_{\mp 2}Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\mathbf{e}) \,. \quad (4.10)$$

This generalises the result for scalar perturbations which involves $\sum_m Y^*_{lm}(\hat{\mathbf{k}})Y_{lm}(\mathbf{e})$ [i.e. is proportional to $P_l(\hat{\mathbf{k}}\cdot\mathbf{e})$].

5 Generation of polarization by Thomson scattering

Linear polarization of the CMB is generated by Thomson scattering of the quadrupole of the temperature anisotropy around recombination. In this section we derive the scattering rate for this process.

To keep things simpler, we shall first discuss scattering of initially unpolarized radiation. Consider scattering from $\mathbf{e}$ into $\mathbf{e}'$, and introduce polarization bases at $\mathbf{e}$ and $\mathbf{e}'$ with the local x (and x') axes in the scattering plane and the local y (and y') axes perpendicular to the plane. Denote the polarization produced by scattering into $\mathbf{e}'$, on the basis adapted to the scattering plane, by $d\bar{Q}(\mathbf{e}')$ and $d\bar{U}(\mathbf{e}')$. Then, if the scattered total intensity is

$$dI(\mathbf{e}') = \frac{3}{16\pi}|d\tau|(1+\cos^2\beta)I(\mathbf{e})d\mathbf{e} \,, \quad (5.1)$$

where $d\tau$ is the increment in optical depth and β is the (scattering) angle between $\mathbf{e}$ and $\mathbf{e}'$, standard results for dipole scattering give

$$d\bar{Q}(\mathbf{e}') = \frac{3}{16\pi}|d\tau|(-1+\cos^2\beta)I(\mathbf{e})d\mathbf{e} \,, \qquad d\bar{U}(\mathbf{e}') = 0 \,. \quad (5.2)$$

Before we can integrate over $\mathbf{e}'$, we must rotate the polarization at $\mathbf{e}'$ to a common basis. We shall use the polar-coordinate basis there, for which $\boldsymbol{\theta}'$ and $\boldsymbol{\phi}'$ are obtained from the scattering-plane x' and y' directions by some rotation $-\gamma'$ in a right-handed sense about $\mathbf{e}'$.[7] The corresponding angle at $\mathbf{e}$ is $-\gamma$. Denoting the scattered polarization on

[7]When discussing the production and propagation of polarized radiation, it is convenient to associate the radiation with the point $\mathbf{e}$ on the sphere. At this point, the propagation direction is outwards so a right-handed polarization basis is formed by $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ there; this is equivalent to using $\hat{\boldsymbol{\theta}}, -\hat{\boldsymbol{\phi}}$ at $\hat{\mathbf{n}} = -\mathbf{e}$ and so the Stokes parameters are the same in both descriptions. However, $(Q+iU)(\mathbf{e})$ is then

the polar-coordinate basis by $dQ(\mathbf{e}')$ and $dU(\mathbf{e}')$, we have

$$\begin{aligned} d(Q \pm iU)(\mathbf{e}') &= e^{\pm 2i\gamma'} d(\bar{Q} \pm i\bar{U}) \\ &= -\frac{3}{16\pi}|d\tau| e^{\pm 2i\gamma'} \sin^2\beta I(\mathbf{e}) d\mathbf{e} \; , \end{aligned} \tag{5.3}$$

To handle the $e^{\pm 2i\gamma'}\sin^2\beta$ factor, we can use the spin-weighted generalisation of the addition theorem for spherical harmonics:

$$\sum_{|m|\leq l} {}_sY^*_{lm}(\mathbf{e})\,{}_{s'}Y_{lm}(\mathbf{e}') = \frac{2l+1}{4\pi} D^l_{ss'}(\gamma, \beta, -\gamma') \; . \tag{5.4}$$

To derive this result, note that we can obtain the triad formed from the scattering-plane x and y directions and the outward normal $\mathbf{e}$ from the global x, y and z frame by applying a rotation $D(\phi, \theta, \gamma)$. Similarly, we obtain the triad at $\mathbf{e}'$ by rotating with $D(\phi', \theta', \gamma')$. However, we can also form the triad at $\mathbf{e}'$ by first rotating the global x, y, z triad with $D(0, \beta, 0)$ and subsequently rotating with $D(\phi, \theta, \gamma)$. It follows that

$$D(\phi', \theta', \gamma') = D(\phi, \theta, \gamma) D(0, \beta, 0) \quad \Rightarrow \quad D^{-1}(\phi, \theta, \gamma) D(\phi', \theta', \gamma') = D(0, \beta, 0) \; . \tag{5.5}$$

We now express the rotations in terms of their matrix representations, $D^l_{mm'}$ and use unitarity to deal with the inverse rotation $D^{-1}(\phi, \theta, \gamma)$. Finally, the relation between the spin-weighted harmonics and the rotation matrices,

$$D^l_{-ms}(\phi, \theta, 0) = (-1)^m \sqrt{\frac{4\pi}{2l+1}}\, {}_sY_{lm}(\theta, \phi) \; , \tag{5.6}$$

establishes the addition theorem in Eq. (5.4).

Armed with the addition theorem, and noting that $\sin^2\beta = \sqrt{8/3}\, d^2_{0\,\pm 2}(\beta)$, we can write

$$\begin{aligned} \sin^2\beta e^{\pm 2i\gamma'} &= \sqrt{\frac{8}{3}} d^2_{0\,\pm 2}(\beta) e^{\pm 2i\gamma'} \\ &= \sqrt{\frac{8}{3}} D^2_{0\,\pm 2}(0, \beta, -\gamma') \\ &= \sqrt{\frac{8}{3}} \frac{4\pi}{5} \sum_{|m|\leq 2} Y^*_{2m}(\mathbf{e})\,{}_{\pm 2}Y_{2m}(\mathbf{e}') \; . \end{aligned} \tag{5.7}$$

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spin +2 and its appropriate expansion in spin harmonics is

$$(Q \pm iU)(\mathbf{e}) = \sum_{lm} (E_{lm} \pm iB_{lm})\,{}_{\pm 2}Y_{lm}(\mathbf{e}) \; .$$

The multipoles E_{lm} and B_{lm} here are related to those in the line-of-sight description by factors of $(-1)^l$ and $(-1)^{l+1}$ respectively and so the parity-invariant power spectra are unchanged.

Integrating Eq. (5.3) over $\mathbf{e}$ is now trivial, and isolates the quadrupole of the incident intensity. Expressing things in terms of temperature anisotropies and temperature-equivalent Stokes parameters, we simply have

$$d(Q \pm iU)(\mathbf{e}') = -\frac{3}{5}|d\tau| \sum_{|m|\leq 2} \frac{1}{\sqrt{6}} \Theta_{2m\ \pm 2}Y_{2m}(\mathbf{e}') \,, \tag{5.8}$$

where $\Theta(\mathbf{e}) = \sum_{lm} \Theta_{lm} Y_{lm}(\mathbf{e})$. Writing $Q \pm iU = \sum_{lm}(E_{lm} \pm iB_{lm})_{\pm 2}Y_{lm}$, we see that the scattered radiation is locally an E-mode quadrupole.

It is left an exercise to generalise the results of this section for scattering of polarized radiation. You should find that the result for the linear polarization is

$$d(Q \pm iU)(\mathbf{e}') = \frac{3}{5}|d\tau| \sum_{|m|\leq 2} \left(E_{2m} - \frac{1}{\sqrt{6}} \Theta_{2m} \right) {}_{\pm 2}Y_{2m}(\mathbf{e}') \,. \tag{5.9}$$

6 Polarization from scalar perturbations

Thomson scattering changes the linear polarization state of the radiation as

$$d(Q \pm iU) = \dot{\tau} d\eta (Q \pm iU) - \frac{3}{5}\dot{\tau} d\eta \sum_{|m|\leq 2} \left(E_{2m} - \frac{1}{\sqrt{6}} \Theta_{2m} \right) {}_{\pm 2}Y_{2m}(\mathbf{e}) \tag{6.1}$$

Here, Θ_{lm} are the multipoles of the fractional temperature fluctuation, $\Theta(\eta, \mathbf{x}, \mathbf{e}) = \sum_{lm} \Theta_{lm}(\eta, \mathbf{x}) Y_{lm}(\mathbf{e})$, and, recall, $\dot{\tau} = -a n_{\mathrm{e}} \sigma_{\mathrm{T}}$. The first term arises from scattering out of the beam and reduces the polarization. The second term, arising from in-scattering, involves the quadrupoles of the temperature anisotropies and the E-mode polarization. In particular, scattering of unpolarized radiation generates linear polarization $d(Q + iU) = \dot{\tau} d\eta \eth^2 \left(\sum_m \Theta_{2m} Y_{2m} \right) / 20$ which is an E-mode quadrupole. Integrating Eq. (6.1) along the line of sight, ignoring reionization and the polarization term in the in-scattering part and assuming instantaneous decoupling, the observed polarization at $(\eta_R, \mathbf{x}_R)$ is

$$(Q \pm iU)(\eta_R, \mathbf{x}_R, \mathbf{e}) \approx -\frac{\sqrt{6}}{10} \sum_m \Theta_{2m}(\eta_*, \mathbf{x}_*)_{\pm 2} Y_{2m}(\mathbf{e}). \tag{6.2}$$

Here, the point $(\eta_*, \mathbf{x}_*)$ lies on the last-scattering surface of the observation event, i.e. $\eta_R = \eta_* + \chi_*$ and $\mathbf{x}_R = \mathbf{x}_* + \chi_* \mathbf{e}$ where χ_* is the distance to last-scattering. Due to spatial inhomogeneities in the temperature quadrupole over the last-scattering surface, the observed polarization will not generally retain the simple E-mode quadrupole character that is induced locally at last-scattering. To see how this works, it is simplest to consider the important cases of density perturbations and gravitational waves separately.

Density perturbations transform as scalars under reparameterisation of spatial coordinates. In Fourier space, this means that every perturbed tensor is azimuthally-symmetric about the wavevector $\mathbf{k}$. The same is true of the temperature fluctuation and so we have

$$\begin{aligned}\Theta(\eta,\mathbf{k},\mathbf{e}) &= \sum_l (-i)^l \Theta_l(\eta,\mathbf{k}) P_l(\hat{\mathbf{k}}\cdot\mathbf{e}) \\ &= (-i)^l \frac{4\pi}{2l+1}\Theta_l(\eta,\mathbf{k}) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\mathbf{e}). \end{aligned} \tag{6.3}$$

Extracting the spherical multipoles gives

$$\Theta_{2m}(\eta,\mathbf{k}) = -\frac{4\pi}{5}\Theta_2(\eta,\mathbf{k}) Y^*_{2m}(\hat{\mathbf{k}}). \tag{6.4}$$

As a concrete example, consider scales that are large compared to the mean-free path l_p. Over one scattering time, a quadrupole temperature anisotropy builds up most efficiently by the Doppler term at the previous scattering. If we locate ourselves at the origin, the temperature anisotropy is

$$\Theta(\eta,\mathbf{0},\mathbf{e}) \sim \mathbf{e}\cdot\mathbf{v}_b(\eta-l_p,-l_p\mathbf{e}) \approx \mathbf{e}\cdot\mathbf{v}_b(\eta-l_p,\mathbf{0}) - l_p\mathbf{e}\cdot(\mathbf{e}\cdot\boldsymbol{\nabla}\mathbf{v}_b)(\eta-l_p,\mathbf{0})\ . \tag{6.5}$$

The quadrupole part of this comes from the last term which, for a single Fourier mode, gives

$$\sum_{|m|\le 2}\Theta_{2m}(\eta,\mathbf{k}) Y_{2m}(\mathbf{e}) \approx \frac{2}{3} k l_p P_2(\hat{\mathbf{k}}\cdot\mathbf{e}) v_b(\eta,\mathbf{k})\ , \tag{6.6}$$

and hence

$$\Theta_2(\eta,\mathbf{k}) \approx -\frac{2}{3} k l_p v_b(\eta,\mathbf{k})\ . \tag{6.7}$$

Note that this is $O(kl_p)$ whereas the monopole source terms, $\Theta_0+\psi$, in the formula for the temperature anisotropy can only produce a quadrupole of $O(k^2 l_p^2)$.

Scattering of the quadrupole generates $Q \pm iU \sim \sum_{|m|\le 2} Y^*_{2m}(\hat{\mathbf{k}})_{\pm 2}Y_{2m}(\mathbf{e})$ locally. If we take $\mathbf{k}$ along z, we have only $m=0$ modes. Using

$${}_{\pm 2}Y_{20}(\mathbf{e}) = \sqrt{\frac{5}{4\pi}} D^2_{0\,\pm 2}(\phi,\theta,0) = \sqrt{\frac{5}{4\pi}}\sqrt{\frac{3}{8}}\sin^2\theta, \tag{6.8}$$

we see that at last-scattering $Q \propto \sin^2\theta$ and $U=0$. How does the character of the observed polarization change by free-streaming through a distance χ_* (ignoring reionization for the moment)? What we observe from this single Fourier mode still has $U=0$, but the angular dependence of Q is further modulated by the spatial dependence of the perturbation over the last-scattering surface. This modulation transfers polarization from $l=2$ to higher multipoles (with most appreciable power appearing at $l=k\chi_*$), but preserves the electric character of the polarization. We can see that this is reasonable by noting that the modulated polarization field has its polarization direction either parallel or perpendicular to the direction in which the polarization amplitude is changing. In more detail, if we take $\hat{\mathbf{k}}$ along the z-axis, the observed polarization at the origin is, from Eq. (6.2),

$$\begin{aligned}(Q \pm iU)(\eta_R, k\hat{\mathbf{z}}, \mathbf{e}) &= \frac{\sqrt{6}}{10}\sqrt{\frac{4\pi}{5}}\Theta_2(\eta_*, k\hat{\mathbf{z}})e^{-ik\chi_*\hat{\mathbf{k}}\cdot\mathbf{e}}{}_{\pm 2}Y_{20}(\mathbf{e}) \\ &= \frac{3}{20}\Theta_2(\eta_*, k\hat{\mathbf{z}})(1-\mu^2)e^{-ik\chi_*\mu}\,, \end{aligned} \tag{6.9}$$

where we have used $Y_{l0}(\hat{\mathbf{z}}) = \sqrt{(2l+1)/4\pi}\delta_{m0}$ and defined $\mu \equiv \cos\theta$. (Note that U is zero in this orientation.) Since the polarization is azimuthally symmetric, the same will be true of any E and B modes. Taking

$$Q + iU = \eth^2(P_E + iP_B)\,, \quad Q - iU = \bar{\eth}^2(P_E - iP_B)\,, \tag{6.10}$$

where $P_E = \sum_{l\geq 2}\sqrt{(l-2)!/(l+2)!}E_{l0}Y_{l0}(\theta,\phi)$ and similarly for ψ_B, we must have

$$(1-\mu^2)\frac{d^2}{d\mu^2}(P_E \pm iP_B) = \frac{3}{20}\Theta_2(\eta_*, \mathbf{k} = k\hat{\mathbf{z}})(1-\mu^2)e^{-ik\chi_*\mu}\,. \tag{6.11}$$

It follows that $P_B = 0$ (the non-zero solutions of $d^2P_B/d\mu^2 = 0$ are combinations of $l = 0$ and $l = 1$ modes and cannot generate polarization) so there are no B modes. Since E and B modes transform irreducibly under rotations, there will be no B modes for a general $\mathbf{k}$. Dropping the proportionality constant, the relevant solution of $d^2P_E/d\mu^2 = e^{-ik\chi_*\mu}$ is $a_0(k\chi_*) + a_1(k\chi_*)\mu - e^{-ik\chi_*\mu}/(k\chi_*)^2$ where a_0 and a_1 should be chosen to remove the $l = 0$ and $l = 1$ modes. Expanding the exponential gives

$$E_{lm}(\eta_R, k\hat{\mathbf{z}}) = -(-i)^l\delta_{m0}\frac{3}{20}\Theta_2(\eta_*, k\hat{\mathbf{z}})\sqrt{4\pi}\sqrt{2l+1}\sqrt{\frac{(l+2)!}{(l-2)!}}\frac{j_l(k\chi_*)}{(k\chi_*)^2}\,. \tag{6.12}$$

Note that the spatial-to-angular projection of scalar polarization is controlled by the same function, $j_l(k\chi_*)/(k\chi_*)^2$, as for the temperature anisotropies from gravitational waves [see Eq. (4.9)].

The multipoles for a general direction of $\hat{\mathbf{k}}$ can be found by the appropriate rotation, giving

$$E_{lm}(\eta_R, \mathbf{k}) = -4\pi(-i)^l\frac{3}{20}\Theta_2(\eta_*, \mathbf{k})\sqrt{\frac{(l+2)!}{(l-2)!}}\frac{j_l(k\chi_*)}{(k\chi_*)^2}Y^*_{lm}(\hat{\mathbf{k}})\,. \tag{6.13}$$

Finally, the E-mode power spectrum evaluates to

$$C_l^E = 4\pi\left(\frac{3}{20}\right)^2\frac{(l+2)!}{(l-2)!}\int d\ln k\left|\frac{\Theta_2(\eta_*, \mathbf{k})}{\mathcal{R}(\mathbf{k})}\right|^2\left(\frac{j_l(k\chi_*)}{(k\chi_*)^2}\right)^2\mathcal{P}_{\mathcal{R}}(k)\,. \tag{6.14}$$

7 Polarization from gravitational waves

For gravitational waves, a temperature quadrupole builds up over a scattering time due to the shear of the gravitational wave:

$$\Theta(\mathbf{e}) \sim -\frac{1}{2} l_p \dot{h}_{ij} e^i e^j \,. \tag{7.1}$$

Consider a circularly-polarized gravitational wave with $\mathbf{k}$ along the z-axis so that the temperature quadrupole is a $|m| = 2$ mode. Equations (4.7) and (6.1) show that Thomson scattering of this temperature quadrupole produce polarization that locally has

$$Q \pm iU \sim l_p \dot{h}^{(p)}(k\hat{\mathbf{z}}) \frac{1}{20} \sqrt{\frac{8\pi}{5}} {}_{\pm 2}Y_{2\,p}(\mathbf{e}) \,, \tag{7.2}$$

where $h^{(p)}(\mathbf{k})$ is the Fourier amplitude of the gravitational wave with $p = \pm 2$ labelling the helicity.[8] Unlike scalar perturbations, the local polarization has both Q and U non-zero and modulating these with the plane wave $e^{-i\chi_* \mathbf{k}\cdot\mathbf{e}}$ will produce both E and B-mode polarization. In detail, the observed polarization is

$$\begin{aligned}
(Q \pm iU)(\eta_R, k\hat{\mathbf{z}}, \mathbf{e}) &\propto {}_{\pm 2}Y_{2p}(\mathbf{e}) e^{-ik\chi_* \cos\theta} \\
&= {}_{\pm 2}Y_{2p}(\mathbf{e}) \sum_L \sqrt{4\pi}\sqrt{2L+1}(-i)^L j_L(k\chi_*) Y_{L0}(\mathbf{e}) \\
&= \sqrt{5} \sum_L \Bigg[\sqrt{2L+1}(-i)^L j_L(k\chi_*) \\
&\quad \times \sum_l \sqrt{2l+1} \begin{pmatrix} 2 & L & l \\ -p & 0 & p \end{pmatrix} \begin{pmatrix} 2 & L & l \\ \pm 2 & 0 & \mp 2 \end{pmatrix} {}_{\pm 2}Y_{lp}(\mathbf{e}) \Bigg] \\
&= -\sqrt{5} \sum_l (-i)^l \sqrt{2l+1} {}_{\pm 2}Y_{lp}(\mathbf{e}) \left[\epsilon_l(k\chi_*) \pm \frac{p}{2} i \beta_l(k\chi_*) \right] \,,
\end{aligned} \tag{7.3}$$

where we have used an addition theorem for the spin spherical harmonics, and the functions

$$\epsilon_l(x) \equiv \frac{1}{4} \left[\frac{d^2 j_l(x)}{dx^2} + \frac{4}{x}\frac{dj_l(x)}{dx} + \left(\frac{2}{x^2} - 1\right) j_l(x) \right] \,, \tag{7.4}$$

$$\beta_l(x) \equiv \frac{1}{2} \left[\frac{dj_l(x)}{dx} + \frac{2}{x} j_l(x) \right] \tag{7.5}$$

follow from substituting the explicit form of the $3j$ symbols and using the recursion relations for the the spherical Bessel functions. Reinstating numerical factors from

[8] A more careful treatment of Eq. (4.4), balancing in- and out-scattering in the presence of the shear source, and properly including the polarization dependence of Thomson scattering, introduces a prefactor of 10/3 in these expressions.

Eq. (7.2), and rotating to a general $\mathbf{k}$, the non-zero E- and B-mode multipoles are

$$E_{lm}(\eta_R, \mathbf{k}) = -\frac{\sqrt{2\pi^2}}{5} l_p (-i)^l \dot{h}^{(\pm 2)}(\eta_*, \mathbf{k}) \epsilon_l(k\chi_*) {}_{\mp 2}Y^*_{lm}(\hat{\mathbf{k}}) \tag{7.6}$$

$$B_{lm}(\eta_R, \mathbf{k}) = \mp\frac{\sqrt{2\pi^2}}{5} l_p (-i)^l \dot{h}^{(\pm 2)}(\eta_*, \mathbf{k}) \beta_l(k\chi_*) {}_{\mp 2}Y^*_{lm}(\hat{\mathbf{k}}) \,. \tag{7.7}$$

We see that gravitational waves project onto both E- and B-mode polarization. The projection from last scattering peaks at multipoles $l \sim k\chi_*$ and divides power roughly equally between E and B. Note also that under a parity transformation, $\dot{h}^{(\pm 2)}(\mathbf{k}) \to \dot{h}^{(\mp 2)}(-\mathbf{k})$. Given that ${}_sY_{lm}(-\hat{\mathbf{k}}) = (-1)^l {}_{-s}Y_{lm}(\hat{\mathbf{k}})$, Eqs (7.6) and (7.7) manifestly give $E_{lm} \to (-1)^l E_{lm}$ and $B_{lm} \to -(-1)^l B_{lm}$, as required, under parity.