Today, the goal is to discuss several theoretical frameworks of the literature in which FC are dynamical, from standard field theory to string theory.

This will give a second motivation for this text: they allow to constraint many theoretical frameworks of gravity beyond GR.

Field Theory

As already explained, if a constant is varying, this means it has to be replaced by a dynamical field.

This has to be done at the level of the action.

And it is a common mistake to let it vary in equations that have been derived under the assumption that this constant was actually constant.

The famous example is VSL (at least in their early version):

They argue

\[ H^2 = \frac{8\pi G}{c^4}\rho - \frac{\kappa}{\alpha^2} + \frac{\Lambda}{3} \]

- Assume e.g. \( G(t) \), \( \alpha(t) \) or \( \Lambda(t) \)
- Then use \( \nabla_{\mu}(\frac{8\pi G}{c^4}\rho + \Lambda g_{\mu}) = 0 \)
can you tell me what is wrong here?

one will get a modified equation of casimir

But there will miss one equation

usually one then postulates $C(t)$ or $G(t)$.

In field theory, this is different

$L(\phi, \cdots, x) \rightarrow \tilde{L}(\phi, \cdots, x)$

Then

1. $\frac{\delta L}{\delta \phi} = \partial_{\phi} \frac{\delta L}{\delta x} \quad$ will end up here in $\partial_{\phi} x$

   so the "sl" equations will be modified

2. $\frac{\delta L}{\delta x} = \partial_{x} \frac{\delta L}{\delta \phi} x$ \quad There is a new equation

   so that there is no freedom to choose $x(t)$ at

hand!

This is important since many wrong papers are

based on this mistake.
Scalar-tensor theories:

Historically, the first "very effective" theory was formulated by Jordan (1937) who considered

\[ S = \int F_g \, d^4x \, \phi \left [ R - \frac{3}{2} (\nabla \phi)^2 - \frac{\phi}{2} F^2 \right ] \]

What would you say about this action?

[to be discussed with students to compare with

\[ S = \int F_g \, d^4x \left [ \frac{R}{2\kappa} - \frac{1}{4\phi} F^2 \right ] \]

\[ \frac{\alpha \cdot G m^2}{\kappa c} \rightarrow (\mu, \xi) \]

1 new d.o.f. \( \phi \)

To understand better the structure of such theories, I concentrate on scalar-tensor theories.

In the Jordan frame, they take the form

\[ S = \int \frac{d^4x}{16\pi G} \left [ F(y) R - g^{00} Z(y) \phi, y \phi, y, y - 2 Y(y) \right ] \]

\[ + S_m \left [ \psi, g_{\mu\nu} \right ] \]

\( G_m \): coefficients of the theory

not to be confused with \( G_n \)

\( (Z, F, U) \): 3 arbitrary functions

only 2 are physical

\( F > 0 \) for gravity to carry mass
Matter fields are all coupled to $g_{\mu\nu}$ minimally

\[ \Rightarrow \text{satisfies the EEP in its weak form.} \]

The equation of motion are obtained by

\[ \delta g_{\mu\nu} \quad \text{[Einstein]} \quad \delta \psi \quad \text{[conservation]} \quad \delta \psi \quad \text{[new eq.]} \]

and are

\[
\begin{cases}
F(y) g_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} + Z(y) \left[ \partial_{\mu} \psi \partial_{\nu} \psi - \frac{1}{2} g_{\mu\nu} (\partial_{\alpha} \psi)^2 \right] \\
+ \nabla_{\nu} \partial_{\nu} \Phi - g_{\mu\nu} \Phi F - g_{\mu\nu} \psi(y) \\
2 Z \delta \psi = F' \Phi - Z' (\partial_{\alpha} \psi)^2 + 2 U' \\
\nabla_{\mu} T^\mu_{\nu} = 0
\end{cases}
\]

with

\[ T^\mu_{\nu} = \frac{\epsilon}{F} \frac{\delta S_{\text{m}}}{\delta g_{\mu\nu}} \]

- if $y = 0$, then we are back to R6 (Einstein + conservation equations)

- $\frac{G_{\mu\nu}}{F(y)} = \text{Jeff}(y)$ it looks like a new $G$ theory

- we see explicitly the difference with QED

G vary in Einstein equation because there are less in $\delta \psi$ and the new (KG) equation.
\[ \nabla_\mu T^{\mu} = 0 \] because of the null energy condition.

This action is usually rewritten in Einstein frame after the change of variables:

\[
\begin{align*}
A(y^a) &= F^{-\frac{1}{2}}(y)
\end{align*}
\]

\[
\begin{align*}
2 V(y^a) &= U(y) F^{-2}(y) \\
g_{\mu} &= F(y) g_{\mu}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{d y^a}{d \phi} \right)^2 &= \frac{g}{4} \left( \frac{d \ln F(y)}{d y} \right)^2 + \frac{1}{2 F(y)}
\end{align*}
\]

so that the action reads

\[
S = \frac{1}{16 \pi G_N} \int d^4 x \sqrt{-g} \left[ R - 2g_{\mu\nu} \partial_\mu \chi \partial_\nu \chi - 4V \right]
\]

\[ + S_m \left( A^2 g_{\mu\nu} ; \psi \right) \]

When we compare to the Jordan frame action, we see:

- \( \chi \) is not directly coupled to \( R \) explicitly.
- \( R G + \chi \) couples to matter to \( A^2 g_{\mu\nu} \).
- Consider a massive scalar field

\[
\text{J.F.: } -\frac{1}{2} \int d^4 x \sqrt{g} \left( g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + m^2 \chi^2 \right) \\
\text{E.F.: } -\frac{1}{2} \int d^4 x \sqrt{g} \left[ A^2(y) g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + m^2 A^2(y) \psi^2 \right]
\]
Consider the action of test particle

\[ \mathcal{S} = \int mc \sqrt{-g} \, dx^0 dx^1 \]

\[ \mathcal{E} = - \int \frac{m \, A(y_0)}{\sqrt{-g}} \, dx^0 dx^1 \]

But

\[ G \, m^2 = \begin{cases} \frac{G \, m^2}{\mathcal{S}} = G \, A^2 m^2 & \text{in } \mathcal{S} \\ G \, m^2(y_0) = G \, A^2 m^3 & \text{in } \mathcal{E} \end{cases} \]

The theory can be thought as

- varying G theory

- varying m theory (with \( \frac{m_0}{m} = c^3 \))

But this is the same theory because the dimensionless quantity

\[ \alpha_G = \frac{G \, m^2}{hc} \]

- test particle follows geodesics of \( \mathcal{S} \) metric but not the \( \mathcal{E} \)
- observables un 2 frames could be distinguished up dimensionless.
Note also that for electromagnetism

\[ S_{\text{maxwell}} = \frac{1}{4} \int \sqrt{-g} \ g^{\mu \nu} g^{\rho \sigma} F_{\mu \rho} F_{\nu \sigma} \ d^4x \]

\[ = \frac{1}{4} \int \sqrt{-g} \ g^{\mu \nu} g^{\rho \sigma} A^{\mu \rho}(\phi) F_{\nu \sigma} \ d^4x \]

In 4 dimensions the action is invariant (conformal invariance)

\[ \Rightarrow \text{null geodesics of } g_{\mu \nu} \text{ and } g^{\mu \nu} \text{ are the same.} \]

Deflection angle is \( \delta \theta = \frac{\mu(\gamma)}{bc^2} \) - careful here!!

- The field equations are

\[
\begin{cases}
G^\mu_\nu = 8\pi \, G \, T^\mu_\nu + 2 \partial_\mu \gamma^\kappa \partial_\nu \gamma^\lambda - g^\mu_\nu \left( \partial_\mu \beta \partial_\nu \beta + \partial_\mu \gamma^\kappa \partial_\nu \gamma^\lambda \right) - 2\gamma^\lambda \partial_\mu \gamma^\lambda \\
\partial_\mu \gamma^\nu \partial_{\nu} \gamma^\kappa = -4\pi G \alpha(\gamma^\kappa) T^\kappa_\nu + V^\kappa_\nu \\
\nabla^\nu T^\kappa_\nu = \alpha(\gamma^\kappa) T^\kappa_\nu \partial_\nu \gamma^\kappa 
\end{cases}
\]

- Einstein equations are the same as in GR

- Eq. of covariant KG are coupled so that we explicitly see the fifth force in this form.

\[ \alpha = \frac{d \ln A}{d \gamma^\kappa} \quad \text{Strength of coupling} \]

\[ \beta = \frac{d \alpha}{d \gamma^\kappa} \]
What is the gravitational constant?

Not $G_n$!

It is dynamical and one needs to define it.

$G_{eff} = G_n A^2(y^o)$ is a guess for the field equation.

The analysis in weak fields leads to the definition of $G_{can}$

$$G_{can} = \frac{G_n A^2}{\alpha_0}$$

$$G_n = G_{can} A^2 \alpha_0 \left(1 + \frac{\alpha_0}{1 + \alpha_0^2}\right)$$

It can be expressed in JF as

$$G_n = \frac{G_n}{F} \left(1 + \frac{F_p^2}{2 F^3 F_p^3}\right)$$

In weak field, one can apply the PPN formalism and derive the deviation from GR by equating the PPN parameter

$$Y_{PPN} = - \frac{2 \alpha_0}{1 + \alpha_0^2} \quad \beta_{PPN} = \frac{1}{2} \frac{\beta_0 \alpha_0^2}{(1 + \alpha_0^2)^2}$$

and the time var. of $G$

$$\frac{\dot{G}_{can}}{G_{can}} = 2 \alpha_0 \left(1 + \frac{\beta_0}{1 + \alpha_0^2}\right) \frac{\dot{y}}{y}.$$
This shows on a particular example that

- Deviation \[ \Delta R \]
- Variations \[ \Delta FC \]

are related and two aspects of the same modification.

Today, in the solar system, \( \Delta \sigma^2 < 10^{-5} \)

we need to be close to GR today but

in principle can be far in the early universe.

*What about other constants*

It seems a pain easy to make a constant dynamical:

- Add a scalar field with kinetic term \( \Delta \phi \)
- Do \( \Delta \phi \rightarrow \Delta \phi (\phi) \)

Let us try with the fine structure constant

\[
S = \int \left\{ \frac{R}{16 \pi \epsilon} - 2 (\Delta \phi)^2 - \frac{1}{4} B(\phi) \right\} F_{ij} F_{ij} \, d^4x
\]

so that

\( \Delta \phi (\phi) = B^{-1}(\phi) \)

We want to emphasize that it has dramatic effects on the UFF.

Indeed, p, n, nuclei have an ETF bound energy so their masses will be size dependent.
As we explained last week, the mass of any nucleus has a contribution

\[
M_A(x) \geq 98.25 \times \frac{Z(2-Z)}{A^{1/3}} \text{ MeV}
\]

Consider sources of point masses

\[
T_{ij}^{\lambda \nu} = \frac{1}{\sqrt{g}} \frac{\epsilon_{ij}^{\lambda \nu} \epsilon_{ij}^{\lambda \nu}}{2} \int \left( \frac{m_i(\psi(x)) u_i^\lambda u_j^\nu}{\sqrt{g}} \right) \delta^\lambda(x-y_i) dx_i \delta^\nu(x-y_j)
\]

The field equation reads

\[
R_{\mu \nu} = 2 \Theta_{\mu \nu} \varphi + 8\pi G \left( T_{\mu \nu} - \frac{1}{2} T g_{\mu \nu} \right)
\]

\[
\square \varphi = -u n G \epsilon \rightarrow d(x) = \frac{1}{\sqrt{g}} \int d\delta \frac{\delta m_i}{\delta \varphi} \delta^\lambda(x-y_i)
\]

\[
= \frac{Z d_i(\varphi) T_i(x)}{L \nu_m T_i}
\]

\[
a_i^\nu = -\delta_i^\nu \left\{ \nabla^\mu \psi \nabla_\mu \psi \left( g^{ij} u_i u_j^k \varphi^k \right) \right\} \text{ (cf I.29)}
\]

It can be shown that the Neumann potential is

\[
V_{neu} = -G \frac{m_i m_j}{r_{ij}} \text{ (cf 2.19)}
\]

So

\[
\frac{d}{dE} \rho_{ij} = \frac{(d_i-d_j) dE}{1 + \frac{1}{2}(d_i-d_j)dE}
\]
\[ \alpha = \frac{\gamma}{\ln \frac{m_{A,0}}{m_{p}}} \]

\[ = \frac{1}{A^{1/3}} \left( \frac{98.25 \text{ Z(eV)}}{A^{1/3}} \right) \frac{dA}{dy} \text{ GeV} \]

\[ m_{A,0} \approx (A-Z)m_{h} + Zm_{p} + 98.25 \alpha \frac{Z(2-1)}{A^{1/3}} \text{ eV} + \ldots \]

[Note: we will have to be more careful because \( m_{p} = \ldots m_{u} \ldots m_{d} \ldots m_{e} \ldots E_{x} \ldots E_{x} \)]

\[ \alpha = 98.25 \left( \frac{\text{eV}}{m_{p}} \right) \frac{Z(2-1)}{A^{1/3}} \alpha(\ln B)_{0} \]

\[ \approx C_{\alpha} \frac{Z(2-1)}{A^{1/3}} (\ln B)_{0} \] with \( C_{\alpha} \approx 98.25 \frac{1}{938.137} \approx 10^{-3} \)

\[ \eta \approx 10^{-6} \int \left[ \frac{Z_{i}, A_{i}, Z_{j}, A_{j}}{Z_{E}, A_{E}} \right] \left[ (\ln B)_{0} \right]^{2} G(10^{-17}, 10) \]

It means that in order to satisfy the bound in the solar system, we need \((\ln B)_{0} \ll 1\)

\[ \text{ie we need to be close to the minimum of the costly function.} \]
This is a generic problem of VFC theories.

Note that we have been considering only the contribution of the E1 binding energy of the nuclei.

In fact we have to include similar contributions for \( e^p \) so that in full generality

\[
M^2_A = \Lambda_{\text{ECD}} \int \left( \prod_{m} \left( \begin{array}{c} m_U \ \ \ m_D \ \ \ m_e \end{array} \right) \right)_{k_1}
\]

so that

\[
\alpha = \frac{a_{\Lambda_{\text{ECD}/p}}}{2} + \sum_{i} \frac{\partial \ln \frac{M^2_A}{2 \Lambda_{\text{ECD}/i}}}{\partial \ln \frac{\Lambda_{\text{ECD}/i}}{\Lambda}}
\]

We shall discuss these contributions later but they are dependent on knowledge of \( \Lambda_{\text{ECD}} \) for the \( \frac{\partial \ln M^2_A}{\partial \ln \Lambda_{\text{ECD}/i}} \) and of the particular charge \( q_i \) for the \( \frac{\partial \ln M^2_A}{\partial \ln \Lambda_{\text{ECD}/i}} \).

\[
\alpha_{(\text{EFC})} \frac{\partial M^2_A}{\partial \Lambda_{\text{ECD}/i}} \text{ are effective charges}
\]

measuring the sensitivity of the mass to \( \Lambda_{\text{ECD}/i} \) of a constant.
This shows that in general we expect a large violation of the universality of the fall.

There are 2 classes of mechanism that allows to pass around this:

1. Lead coulping mechanism (Dynam. Northward)
2. Chameleon-like

I will describe them at the end of the lecture.

Motivation from high-energy physics

Historically one motivation arises from extra-dimensions.

Consider a 5D Einstein-Hilbert action

\[ S_5 = \frac{1}{12\pi^2 G_s} \int \sqrt{-g} \, d^5 x \]

and decouple the 5-dimensional metric as

\[ \overline{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \frac{A_{\mu} A_{\nu}}{M^2} \phi^2 & \frac{A_{\mu}}{M} \phi^2 & \mathbb{1} \\ \frac{A_{\nu}}{M} \phi^2 & \phi^2 & \mathbb{1} \end{pmatrix} \]

\[ [A_{\mu}] = M \]

So the 15 dof of \( \overline{g}_{AB} \) are decoupled in

10 + 4 + 1 dof

\[ X^A = (x^\mu, y) \]
Kaluza cylinder condition: the theory is compactified on a circle.

It assumes that nothing depends on \( y \) \( \text{i.e.} \partial_y x = 0 \)

\[
X(x^\mu, y) = \Sigma \chi_\nu(x^\mu) e^{2\pi i \frac{\nu}{L} y}
\]

This means we can, for scales \( \gg L \),

scale of compactification

i.e. At low energy we can neglect \( n \neq 0 \) modes.

The action \( S_5 \) can be rewritten in terms of \( (g_{\mu\nu}, \phi, A_\mu) \) as

\[
S = \frac{1}{16\pi G} \int d^4 x \sqrt{g} \phi \left( R - \frac{\phi^2}{4\pi^2} + \frac{\partial^\mu F_{\mu\nu} \cdot \partial^\nu F_{\nu\mu}}{4} \right)
\]

with \( G = \frac{3\pi G_5}{4V(5)} \) \( V(5) = \int dy \).

As we discussed earlier this is a theory with varying \((g, \phi)\) if \( A_\mu \) is the U(1) of the standard model.

Note that there is no term with \( \phi \) alone since

\[
R \nabla^2 \phi \neq 0 \quad \text{but} \quad \sqrt{g} = \phi \sqrt{g} \quad \text{so that we have} \quad \nabla^2 \phi, \text{ i.e. a total derivative.} \]
If needed, one can use the far of the Rucci tensor
\[ \bar{R}_{\mu} = \frac{1}{24} \, \phi^4 \, F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \left( \partial \phi \partial \phi + F_{\mu \alpha} F^{\mu \alpha} \right) \phi \]
\[ \bar{R}_{\mu \nu} = \frac{1}{4} \left[ \bar{R}_{\mu} A_{\nu} + \frac{1}{2} \phi \left( \nabla^2 F_{\mu} + 3 \frac{\partial \phi}{\phi} F_{\mu} \right) \right] \]
\[ \bar{R}_{\mu \nu} = \bar{R}_{\mu \nu} - \frac{1}{2} \, \nabla_{\mu} \nabla_{\nu} \phi - \frac{\phi^2}{24} \, \nabla^2 F_{\mu} F_{\nu} \phi + \frac{1}{2} \left( A_{\mu} \bar{R}_{\nu \lambda} + A_{\nu} \bar{R}_{\mu \lambda} \right) \]
\[ - \frac{1}{24} \, \bar{R}_{\mu} A_{\nu} A_{\mu} \]

- Generically, any theory with extra dimensions will reduce to an effective 4D theory with gauge

- In the simplest form, one expects an updated

- In particular \( \phi = 0 \) is not a solution of the field equations

\[
\begin{aligned}
\Box \phi &= \frac{\phi^3}{12} \, F_{\mu \nu} F^{\mu \nu} \\
\nabla^2 F_{\mu \nu} &= - 3 \frac{\partial \phi}{\phi} F_{\mu \nu} \\
G_{\mu \nu} &= \frac{\phi^2}{24} \, T^{(\phi)} + \frac{1}{2} \left( \nabla \partial_{\mu} \phi - g_{\mu \nu} \partial \phi \right)
\end{aligned}
\]

unless \( A_{\mu} = 0 \).

So one cannot recover Einstein + Maxwell

If \( A_{\mu} = 0 \), which a solution of the field equation, then we recover a scalar-tensor theory with no potential

Note that usually, the scalar field is redefined as
\[ \phi = e^{\frac{\sqrt{n}}{n} \frac{\sqrt{\phi}}{24}} \quad \text{and} \quad \phi \in \text{the dilaton with } [\phi] = 11 \]
Note: Even if \( \phi \) has no explicit kinetic term in the action, it is a dynamical field, as can be seen from equation \( D\phi = \ldots \).

One can, as in ST Higgses, diagonalize the d.o.f. by switching to Einstein frame. One performs

\[
S = \frac{\Lambda}{12\pi^2 G_{\text{S}}} \int \sqrt{g} R
\]

\[
= \frac{\Lambda}{16\pi G_{\text{N}}} \int \sqrt{-g} \left( R - \frac{\phi^2}{4\pi^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \frac{\phi^2}{\phi_2^2} \partial_{\mu}\phi \partial^\mu\phi \right)
\]

so that \( \phi \) behaves as a canonical field.

It follows that (if \( A_\mu \) is the electromagnetic \( A_\mu \))

\[
\alpha^S = \frac{1}{4\pi} e^{-\frac{4\pi}{137} \alpha_2 \phi^2}
\]

Note that \( \alpha^S \sim \frac{1}{137} \) if \( \phi \sim 10^2 \Lambda_4 \)

but \( \phi \) dynamical so is \( \alpha \) and one needs to explain this coincidence.

In the case of Dextra-dim, \( G = \phi^{-D} \), \( \alpha_i \sim \kappa_i(\phi) \phi^{-2} \).
Donny the compactification, we have a typical mass scale
\[ \Pi \sim R^{-4} \]

If \( \Pi \) is large enough, low energy observables are not sensitive to the fifth dimension.

At higher energy, there appear a series of excited modes.

Consider a scalar field
\[ S_{\Sigma} = -\frac{1}{2} \int d^5x \sqrt{g} \, g^{\mu \nu} \partial_{\mu} \Sigma \partial_{\nu} \Sigma \]

\[ \Sigma(x^\mu, y) = \sum_{n=0}^{\infty} \delta_n(x^\mu) e^{i n y} \phi \]

and expand the action (assuming \( \partial_\phi = 0 = \partial_y \phi \), i.e., consider only the zero modes of \( g^{\mu \nu} \)). Then, we find

\[ S_{\Sigma} = -\frac{\alpha_1}{\Pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\phi}{\phi^2} |\delta \phi|^2 \right\} \]

By renormalizing \( \delta \phi \) as \( \delta \phi = \frac{\pi}{\alpha} \delta_n \) we obtain the action for

- massless field \( \phi_0 \)
- series of massive fields \( \phi_n \) with masses \( m_n = \frac{\alpha \phi_0^2}{\phi^2} \)

So the masses depend explicitly on the dilaton and are thus dynamical.
Some motivations from string theory

String theory is based on the idea that fundamental objects are no more point particles but 1-dimensional objects.

Superstring theory including bosons and fermions is found to be consistent in $D=10$.

$\Rightarrow$ need of 6 spatial extra dimensions

So Kaluza-Klein effects, including dynamical constants, are generic of string theory.

There exist different theories, only 5 are consistent to which one needs to add 11-d supergravity

\[ \text{Type I} \quad \text{IIA} \quad \text{IIB} \]
\[ \text{11D-Sugra} \quad \text{M-theory} \quad \text{F-theory} \]
\[ \text{E}_8 \times \text{E}_8 \quad \text{SO}(32) \]

Strings can be of 2 types: open and closed

There exist dualities between these theories.

My goal is indeed not to describe these theories but to have a look at their low energy effective limit.
For type IIA heterotic string theory:

They contain closed string and the tree-level action

\[ S_H = \int d^{10}x \sqrt{-g_{10}} \, e^{-\frac{2\phi}{\kappa}} \left\{ \pi_H^2 \left( \phi_{10} + k \phi \right) \right\} - \frac{\pi_H^6}{\kappa} F_{\mu
u} F^{\mu\nu} + \ldots \]  

This action needs to be compactified. Assuming a 6D Kalabi-Yau space, it leads to

\[ S_H = \int d^6x \sqrt{-g_6} \, e^{-\frac{2\phi}{\kappa}} \left\{ \pi_H^2 \left[ \phi_{10} + \left( \frac{\nabla \phi}{\kappa} \right)^2 - \frac{1}{6} \left( \frac{\nabla \phi}{\kappa} \right)^6 \right] - \frac{\pi_H^6}{\kappa} F_{\mu\nu}^2 + \ldots \]  

\[ \text{with} \quad \phi = \varphi_6 e^{-2\phi/\kappa} \]

so we have

\[ H^2 = \pi_H^2 \phi \quad g_{7/4}^2 = \pi_H^6 \phi \]

if the matter fields (in the d.o.s.) are minimally coupled to \( g_4 \).

For type I:

it contains open strings and its low energy action is

\[ S_I = \int d^{10}x \sqrt{-g_{10}} \, \pi_H^6 e^{-\frac{2\phi}{\kappa}} \left[ -\frac{1}{8} \pi_H^2 \pi_H^2 \phi_{10} - \frac{F_{\mu\nu}^2}{\kappa} + \ldots \right] \]

terms, describe the

dynamics of the dilaton

and fermions.

contrary to \( S_H \), the dilaton couples differently

to \( \phi \) and \( F^2 \)

Type I requires D9-branes to be consistent.
After compactification, one gets at tree level

\[ S_I = \int d^{10}x \sqrt{-g} e^{-\phi} R_{10} - \int d^4x \sqrt{-g} e^{-\phi} \frac{1}{4} F^2. \]

"gravity sector"

"Y-fields localized on the brane,"

This 2 examples shows that at tree level, the existence of a dilaton implies that effective 4D constants are dynamical.

- Effect of loop corrections: include KK excitations and one gets concludes

\[ g_{\gamma \gamma}^{-2} = \Pi_{a}^6 \phi - \frac{b_0}{2} (\Pi_{a} R)^2 + \ldots \]

and the coupling is no more universal.

- At tree level, there is no stabilisation mechanism

  \[ \text{why are the constants so constant?} \]

- Witten (1987), in string theory all dimensionless constants are dynamical.
A large set of models

These theoretical motivations have motivated the formulation of a series of phenomenological models. I just summarize some of them.

**Belavkin model** (1982)

It introduces a modification of the electromagnetic sector by introducing a scalar field $\phi$ and such that

$$e_i = e_0; \phi(x^i)$$

Its dynamics is described by

$$S_\phi = -\frac{ke}{2l^2} \int \frac{g\mu_0 e\phi \partial e\phi}{e^2} \sqrt{g} d^4x$$

The electromagnetic sector is modified by

$$e_i A_\mu \rightarrow e_i \phi A_\mu$$

The gauge invariance is preserved if $\epsilon A_\mu \rightarrow \epsilon A_\mu + \partial_\mu \nu$

It follows that the action for $A_\mu$ is

$$S = -\frac{1}{16\pi} \int F_{\mu\nu} F_{\mu\nu} \sqrt{g} d^4x$$

$$F_{\mu\nu} = \frac{1}{\xi} \left[ (\epsilon A_\nu)_\mu - (\epsilon A_\mu)_\nu \right]$$
The gravitational sector is described by the standard Einstein-Hilbert action.

\[ S = \frac{c^3}{16\pi G} \int R \sqrt{g} \, d^4x - \frac{1}{16\pi} \int \frac{e^{-2\phi}}{\sqrt{g}} \, d^4x - \frac{1}{8\pi G} \int \left( \nabla \phi \right)^2 \sqrt{g} \, d^4x \]

\[ \kappa = \frac{1}{(4\pi \hbar c)} \int \nu = \int d\mu \nu \]

Under this form the structure of the theory is clear and our previous discussion shows that it will be untrouble with. UFF

The model can then easily generalized to allow the couply of \( \phi \) to other fields.

\[ S = \frac{1}{2} \Pi^{\mu}_{\lambda} \int R \sqrt{g} \, d^4x - \int \left[ \frac{1}{2} \Pi^{\mu}_{\lambda} \gamma^{\nu}_{\lambda} \, d\mu + \frac{1}{4} B_\mu \left( g^{\mu \nu} F^2 \right) \right] \sqrt{g} \, d^4x \]

\[ + \int \left\{ \sum N_i \left( \partial \phi - m_i \right \} B_\mu \gamma_{\lambda} \right\} \nu_i + \frac{i}{2} \xi^\alpha \partial \phi \]

\[ \nabla = \gamma_{\mu} \left( \partial - i e A_\mu \right) \]

\[ D = \gamma_{\mu} \left( \partial - i e A_\mu \right) \]

\[ \Lambda \rightarrow \phi \text{ or other potential} \]
The models of free junctions $B_L$, $B_N$, $B_V$, $B_X$ usually can be extended as

$$B_X = 1 + \xi_X \phi + \frac{1}{3} \xi_X \phi^2$$

for instance

$$\Delta x = \frac{e^2}{4\pi B_0(\phi)} \implies \Delta a = \xi_F \phi + \frac{1}{2} (\xi_F - 2 \xi_X) \phi^2$$

It is a very general framework that includes Behr, Brevi, Briche...

The dynamics is modified cosmologically by the coply to $\Lambda$ CDM

If Bi has a minimum then it can dynamically lower the evolution of UCC.

**Strong inspired models:**

The other phenomenological framework was proposed by Damour and Dine.

Starting from the tree-level actions of string theory, they argued that the full string loop expansion should lead to an action of the form
\[ S = \int d^4x \sqrt{-g} \left[ \Pi^2 \{ B_y(\phi)^2 + 4B_y(\phi) [B_\phi - (\nabla \phi)^2] \right. \\
\left. - B_y(\phi) \frac{k}{4} F^2 - B_y(\phi) \bar{\psi} \gamma^\mu \gamma_\mu \psi \right. \\
\left. + \cdots \right] \\

B_y, B_\phi, B_\psi, B_y \ldots \text{ are not known functions of } \phi. \\

Assuming they are obtained from a genus expansion, in the limit \( \phi \to \infty \) they shall be of the form \\

\[ B_x(\phi) = e^{-2\phi} + c_{(0)} e^{2\phi} + \cdots \]

Again, one can shift to the Einstein frame by \\

\[ g_{\mu\nu} = c^{-1} B_g f_{\mu\nu} ; \ y = \sqrt{-c B_g} B_y^{\frac{1}{2}} \phi \]

\[ \phi = \int \left[ \frac{2}{3} \left( \frac{B_g}{B_y} \right) - \frac{2 B_y^{\frac{1}{2}}}{B_g^{\frac{1}{2}}} + \frac{2 B_y^{\frac{1}{2}}}{B_g^{\frac{1}{2}}} \right] d\phi' \]

so that

\[ S = \int \frac{d^4x \sqrt{-g}}{16\pi G} \left[ R - 2\phi^2 - \frac{k}{4} B_y(\phi) F^2 - \bar{\psi} \gamma^\mu \gamma_\mu \psi \right. \\
\left. + \cdots \right] \\

\[ g_{\mu\nu}^{-1} = k B_y(\phi) \]

Thus

\[ g_{\mu\nu}^{-2} = k B_y(\phi) \]

This is a very general framework that allows to discuss the conditions on the functions \( B_x \) so that one avoids problems with the UFF.

I shall come back on this later after a
Discussion on the coupled variation of constants.

Least coupling mechanism

Let me come back on pure scalar-tensor theories.

The equation of evolution in Einstein frame for $\phi$ is

$$ \Box \phi = -\rho T G \alpha(\phi) T + V' $$

Assume

$$ V = 0 \quad \text{(massless dilaton)} $$

$$ A = e^{a(\phi)} \quad a = \frac{1}{2} \beta \phi^2 $$

Consider the dynamics in a FL space-time

$$ ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j $$

Then

$$ \begin{cases} 
\dot{\phi} + 3H \phi = -\rho T G \alpha(\phi) (\rho - 3P) \\
\rho + 3H (\rho + P) = \alpha(\phi) (\rho - 3P) \frac{d\phi}{dt}
\end{cases} $$
Assume \( P = \psi P \)

\[ \mu_x \quad P = -\ln (1+2) \quad \text{as line variable} \]

The KG equations rewrite as

\[
\frac{2}{3 \cdot \psi^2} \quad \frac{\psi''}{m(\psi)} + (1 - \omega) \psi' = -a(\psi) (1 - 3 \omega) \quad \uparrow \quad \text{de-pug} \quad \uparrow \quad \text{potential}
\]

If \( \beta > 0 \) and \( \omega \neq \frac{1}{3} \), \( \psi \to 0 \)

\( \uparrow \) During RPU, \( 1 - 3 \omega = 0 \), \( \psi \to c_t \)

Is it always so? \( \cdots \) Turn threshold:

\[ \text{No} \]

\( \uparrow \) Smaller era: The source term is active

There exists an exact solution but we can discuss what happens on a plot

- Dependence of \( \beta \), \( \omega \) chm.

[Slide 3-5]

We shall later discuss how it generalizes to the general models
Chameleon

Another interesting idea is to play with the potential of the field.

\[ \square \phi = - \frac{d V_{\text{eff}}}{d \phi} \quad V_{\text{eff}} = V(\phi) + A^2(\phi) \rho \]

Assume \( V(\phi) \approx \ln A \phi \)

\[ A^2 \approx e^{\phi / \rho} \]

then \( V_{\text{eff}} \) has a minimum that depends on the local matter density!

\[ \Rightarrow \text{field light in space} \]

massive on Earth

if it is massive enough then one may avoid laboratory constraints.
Coupled Variations of the Constants

- Gauge coupling unification

In quantum field theory, the computation of scattering processes include higher order corrections of the coupling constants related to loop corrections, that introduce ultraviolet divergences.

Depending on the theory, these integrals may be finite, log a power law divergent in terms of a UV cut-off.

Renormalizable: physical quantities do not depend on the choice of the cut-off.

But may depend on the energy $E$ of the process.

The renormalization group allows to compute the dependence of a coupling constant as a function of $E$

$$\frac{d \ln q_i(E)}{d \ln \epsilon} = \beta_i(E)$$

depend on gauge group particle content.

In the standard model

$$\beta_i(g_i) = \frac{9 \alpha_i^2}{4 \pi^2} \frac{5 n_f}{9}$$

number of generations of fermions
\[
\beta_2(q_2) = -\frac{g_2^3}{\pi^3} \left( \frac{11}{6} - \frac{9}{8} \right)
\]

\[
\beta_3(q_3) = \frac{g_3^3}{\pi^2} \left( \frac{11}{6} - \frac{11}{6} \right)
\]

At large momentum, couplings 2-loop correction become less and less important. 

\[\alpha_3(q) \to \frac{6\pi}{(33 - 3\ln q)}\]

\(\Lambda_{\text{QCD}}\) is the scale at which \(\alpha_3\) diverges.

**Slide 6**

> Idea that the 3 non-\textit{gauge} couplings may have a single common coupling strength at high energy.

During idea of Grand Unified Theories

\[\text{SO}(10) \times \text{SU}(5) \to \text{SU}(3) \times \text{SU}(2) \times U(1)\]

It has 2 consequences for us:

- There exists relations between the yukawa couplings.
- \(\Lambda\) scale \(\mu\) such that

\[\alpha_1(\mu) = \alpha_2(\mu) = \alpha_3(\mu) = \alpha_4(\mu)\]
The variation of $a$ has to be accompanied by a variation of the couplings.

From the renormalisation group equation, one gets

$$a^{-1}(E) = a^{-1}(E_0) - \frac{b_i^*}{2\pi} \ln \frac{E}{E_0}$$

with

$$\begin{pmatrix}
\frac{11}{10} & -\frac{19}{8} & 7 \\
\frac{33}{5} & 1 & -3
\end{pmatrix}$$

the standard model

and

$$\begin{pmatrix}
\frac{33}{5} & 1 & -3
\end{pmatrix}$$

the $\mathcal{N}=1$ SUSY

$b_i$ depends on the particle content and jumps when there is a mass threshold.

**e.g.**

$$E \rightarrow M_{th} E_f$$

$$a^{-1}(E_f) = a^{-1}(E_0) - \frac{b_i^*}{2\pi} \ln \frac{E_f}{E_0} - \frac{b_i^*}{2\pi} \ln \frac{M_{th}}{E_f}$$

This implies that the running depends on the mass spectrum.

For $d_3$; for $E > M_4$ one has

$$\Lambda_{\alpha\beta} = E \left( \frac{m_c m_b M_4}{E} \right)^{2/12} \varepsilon^{-3/8} d_3(E)$$

Thus $\Lambda_{\alpha\beta}$ shall also vary.

$m_i = \tilde{h}_i V$ so that variation of Yukawa $2 \Lambda_{\alpha\beta} \lambda$ is correlated.
We can deduce that

\[
\frac{\Delta N_{\text{acc}}}{N_{\text{acc}}} = R \frac{4 \pi^2}{27} \left( \frac{3}{v} \frac{\Delta r}{\overline{r}} + \frac{\Delta e}{\overline{e}} + \frac{\Delta b}{\overline{b}} + \frac{\Delta h}{\overline{h}} \right)
\]

\[R = \frac{\delta N_{\text{acc}}}{\delta \ln v}\]

This number depends
- on GUT model
- on particle content

Typically \( R = 30 - 50 \)

In some models, one assumes a relation between \( v \) and \( \overline{h} \) (dimensional transmutation models) as

\[ v = M_p e^{-\frac{g_{tr}}{\overline{h}}} \approx O(1) \]

so that weak scale is related to \( M_p \).

One can also define

\[ S = \frac{\delta \ln v}{\delta \ln h} \quad \text{Typically} \quad S \approx 160 \]
The situation is thus as follow

\[ \{ \alpha_i, h, v, \Lambda_{\text{car}} \} \rightarrow \{ \alpha; v \} \quad (R,S) \]

model-dependent.

There is a large literature on the copulation of these numbers (see Refs in the review).

What we shall recall:

- in most GUT and realistic models, one cannot just set it constant to vary.
- the correlations are model-dependent.

We will come back on that later but it has important consequences:

- in the chiral limit \( m_p = \Lambda_{\text{car}} (1 + 6 \left( \frac{m_1}{\Lambda_{\text{car}}} \right) \)
- \( m_e = h v \)

\[ \Rightarrow \frac{\delta \rho}{\rho} = -0.8 R \frac{\delta \alpha}{\alpha} + 0.6 (5+1) \frac{\delta h}{h} \]

both \( \frac{\delta \alpha}{\alpha} \) and \( \frac{\delta h}{h} \) are in principle observable

- a cross-check if \( m_e \) is not small
- allow to constrain \( R \), i.e. window on only models.
Masses

The local tests of QFT involve macroscopic objects.

We need to compute $m(A, Z)$ or $m_p$ in function of the fundamental constants.

We start from:

$$m(A, Z) = Z m_p + (A-Z) m_p + Z m_e + E_s + E_{\text{En}}$$

2 difficulties:

- $\frac{S m_p}{m_p}$, $\frac{S m_n}{m_n}$
- $E_s$

→ Link between QCD and nuclear physics.

There are a lot of phenomenological models:

- Chiral perturbation theory
- Non-relativistic constrained quark model
- Combinations.
For $E_{En}$, we still use Bethe-Heitler formula

$$E_{En} = 98.25 \frac{Z(2-Z)}{A^{1/3}} \text{ MeV}$$

Then

$$m_{(p,n)} = m_u + b_{(u,d)} m_d + B_{(p,n)} q$$

$\uparrow$ nuclear binding energy

So

$$m_{A,Z} = (A m_u + E_S) + (Z b_u + N b_d) m_u$$

$$+ (Z b_d + N b_u) m_d$$

$$+ (Z B_p + N B_n + 98.25 \frac{Z(2-Z)}{A^{1/3}} \text{ MeV}) d$$

$$+ Z m_e$$

$B_n$ and $B_p$ can be computed (Here is a full Phys.Rept on that!)

$$B_p \alpha = 0.63 \text{ keV} \quad B_n \alpha = -0.3 \text{ keV}$$

Note: Even if one assumes only $B_p \phi P^2$, then

it induces a coupling to $p, n$ in the action

and cosmologically to $p_b$. This cannot be avoided.

[Many mistakes here!]
\[
S = \left(\frac{1}{2} \partial \phi + \frac{1}{2} (\partial \phi)^2 + V(\phi) + \frac{1}{4} B_\phi(\phi) \partial^2 \phi + \sum \bar{\psi}_j \Gamma_j \psi_j - m_j \bar{\psi}_j \psi_j \right) F \, d^4x
\]

\[
B_\phi(\phi) = e^{\frac{\phi}{\mu}}
\]

Because of the radiative coupling, it induces a coupling to nucleons:

\[
\bar{N} = m_N^{-1} \langle N | \left( \frac{\mu}{4} \right) F_{\mu \nu} | N \rangle
\]

\[
\frac{\epsilon_{1F}}{\mu} = \frac{0.0007 \epsilon_{1F}}{\mu}
\]

\[
\epsilon_{1F} = 0.00015 \epsilon_{1F}
\]

\[
\frac{\epsilon_{1F}}{\mu} = 0.0007 \epsilon_{1F}
\]

\[
\frac{\epsilon_{1F}}{\mu} = 0.00015 \epsilon_{1F}
\]

\[
\Rightarrow \begin{cases} 
\text{masses are the dependent} & \text{(Shrawan et al.)} \\
\text{dynamics is modified} & D \phi + m_\phi^2 \phi = B\phi(\phi) \phi
\end{cases}
\]

We now need to determine \( E_5 \):

- I will describe deuterium in BBN (easier because simpler nuclei).

- For larger nuclei, there is no consensus in the literature.

- Liquid drop model

\[
E_5 = \frac{A}{A} \left( a_V - \frac{A}{A^{1/3}} - a_A \frac{(A - 2 \lambda)^2}{A^2} \right) + q_p \frac{\lambda^A_0 + \lambda^A_0}{A^{3/2}}
\]

\[(a_V, a_A, q_p) = (15.7, 12.8, 23.7, 11.2)\]
- Damour-Donoghue ( chiral approach )

\[ E_s = q_3 + \frac{b_3}{A^{1/3}} \quad \text{with} \quad q_3 = q_3^{\text{chiral}} + m_\pi^2 \frac{-a_3}{6m_N^2} \]

- Fitting family derived for effective field theories (semi-empirical)

\[ \frac{E_i}{A} = - \left( 120 + \frac{97}{A^{1/3}} \right) \frac{1}{3} + \left( 67 - \frac{33}{A^{1/3}} \right) \frac{1}{V} \]

- scalar (almost) \quad - vector (repuls) \quad - nuclear (at least)

---

The order of magnitude are similar but there is a dependence of the number due to the model dependence and to the difficulty to relate QCD to nuclear physics.

---

Two applications

1. For the universality of free fall, we define

\[ \frac{\Delta h}{\Delta t} \times \frac{\Delta m}{\Delta t} \]

These tests can be educated by the following letters.

Assuming drop, we have the values in the slide 7 for the shaded experts.

2. In the strong dilaton model, the field \( \phi \) can couple to the changes

\[ \alpha', \quad B = N+2, \quad D = N-2 \quad \text{and} \quad E = \frac{Z(2-Z)}{A^{1/3}} \]
If all $B$ have the same minimum then

$$I_{BG} \propto (\phi_0 - \phi_m)^2 \left[ c_6 \delta \left( \frac{E}{E_0} \right) + c_0 \delta \left( \frac{E}{E_0} \right) + c_8 \delta \left( \frac{E}{E_0} \right) \right]$$

But if they do not have the same minima we expect large variation of $\phi \to \sigma$ constraint on the stabilisation mechanism.
To finish, I want to highlight the links with dark energy and discuss and alternative model.

1. Dynamical DE models utilize a $\phi$ slow roll.

If it also sources van. of $F_\phi$ then

$\dot{\phi}$ is related to $\ddot{\phi}$ and thus to $\phi(t)$

From $w_\phi = \frac{\rho}{\rho} \Rightarrow \phi = H \sqrt{3\Omega_\phi (1 + w_\phi)}$

This implies that constraints on $\dot{\phi}$ set constraints on $(w_\phi, \Omega_\phi)$ even when $\rho_{\phi} < \rho_{m} \text{ i.e. at large redshift.}$

Also, in all these model, $(w_\phi, \Omega_\phi)$ gives $\dot{\phi}$ and $\phi$ so that the local shall not be forgotten.

2. In all the previous models, $\phi$ is light so that line variation are larger than space variations.

Can we construct a model in which this is not the case

$\Rightarrow \text{yes}$ olve-peloso-jppu.

This what I discuss to finish this lecture on the theoretical toolbox.
The theory is very similar to those considered today

\[
S = \int \left[ \frac{1}{2} \nabla \phi^2 - \frac{1}{2} (\partial \phi)^2 + V(\phi) + \frac{1}{\mu^2} R \phi(\phi) F_{\mu \nu}^2 - \sum_i \bar{\psi}_i D \psi_i - B_i(\phi) m_f \bar{\psi}_i \psi_i \right] \sqrt{-g} \, d^4x.
\]

and we choose the potential \( V = \frac{1}{4} A (\phi^2 - \mu^2)^2 \)

and the couplings \( B_i(\phi) = e^{\frac{\phi}{\mu}}\frac{\phi}{N_\phi} = 1 + \frac{\phi}{\mu} \frac{\phi}{N_\phi} \)

Parameters of the model \((\lambda, \mu, \eta, \xi_p, \xi_i)\)

Assume \( \xi_i = 0 \). Still there is a coupling to proton

(you remember why?) \( \Rightarrow \xi_p, \xi_f \neq 0 \)

The evolution of \( \phi \) is dictated by the KG equation

\[
\Box \phi + \frac{\partial V_{eff}}{\partial \phi} = 0
\]

\( V_{eff} \) has 3 contributions

- coupling to \( F^2 \) and thus to \( p_6 \)
- loop corrections that scale as \( \frac{\mu^2}{\Lambda^4} \frac{\phi^2}{\mu^2} \)
- finite \( T \) corrections had scale as \( \frac{\phi^2}{\Lambda^4} \frac{d^2 \phi}{d \phi^2} \)
Note that there is no coupling to \( \Phi \) because \( \xi_\Phi = 0 \).

\[
V_{\text{eff}} = V(\phi) + \frac{\xi_\phi}{2N} \partial^2 \phi \partial \phi + \frac{\xi_\phi^2}{2} \phi^2 \partial^4 + \frac{1}{4} \phi^2 \partial^2
\]

In order to have \( \frac{\xi_\phi}{\lambda} \sim 10^{-6} \) we need

\[
\frac{\xi_\phi}{\lambda} \sim 2 \frac{\lambda}{2} \frac{1}{\xi_\phi} \sim 10^{-6}
\]

\[
\phi \approx \phi_0 \sim 2 \Phi
\]

we assume \( H \sim \eta \) and \( \xi_\Phi \sim 10^{-6} \).

\[
\text{Squall} \sim \frac{\text{Squall} \eta}{\Phi_0} \sim \left( \frac{1}{10^6 \text{eV}} \right)^3 \Rightarrow \text{arise} \quad \eta \sim 6 \text{ (eV)}
\]

\[
\eta \sim 1; \eta \sim 1 \text{ keV}; H \sim 1 \text{ eV}^2; \eta \sim 1
\]

\[
\xi_\Phi \sim 10^{-6} \text{ eV}^2 = 10^{-6} \xi_\Phi
\]

Such a theory would be a called field theory with \( \frac{H}{\xi_\Phi} \sim 10^6 \text{ eV} \)

(this is the mass scale entering the only due a full part of the theory)

The phase transition takes place around \( \Phi_0 ^2 = 1 \text{ keV} \), i.e. 6 order of magnitude lower.
Using the full potential one can just estimate $T_c$ one finds $\frac{T_c}{T_0} = 1 + 2f \sim 8.5 \cdot 10^9$

- CMB constraints: integrated SW

$$\frac{\delta T}{T} \sim G U H^{-1}$$

$$\frac{\delta T}{T} \sim 10^{-6} \left( \frac{1}{10^9} \right)^3$$

- Astrophysical constraints.

Even though $\phi$ is heavy, $M_{\phi} > 1$ TeV it can be produced in SN

$$\Gamma \sim \frac{3f^2 T^3}{M_{\phi}^2}$$

This could result in an excessive rejection.

But $\phi$ decays in $2 \chi$ with

$$\Gamma \sim \frac{3f^2 M^3}{10^9} \sim 3 \cdot 10^{-16}$$

we can show that if decay within the core space

$$\frac{M_{\phi}}{10^9} \gg 6 \left(10^{-7}\right) \left(\frac{10^{-6}}{f^2}\right)$$

- Stability

  ![Diagram of potential with $V_{eff}$ vs $\phi$](attachment:diagram.png)

Due to $Bf$,

$\phi$ life is $\Rightarrow$ age of universe.
This is a good example of all the cases one needs to take when playing this game.

**SUMMARY:**

- We have described a tool box to conduct theories with dynamical systems.
- We have emphasized the importance of URF.
- We have described motivations from string theory.
- We have seen ideas about the connections between constants in unification models.
- We have a model of pure spatial variation.

Next, we shall discuss all physical systems to put constraints on the variations of Fc.