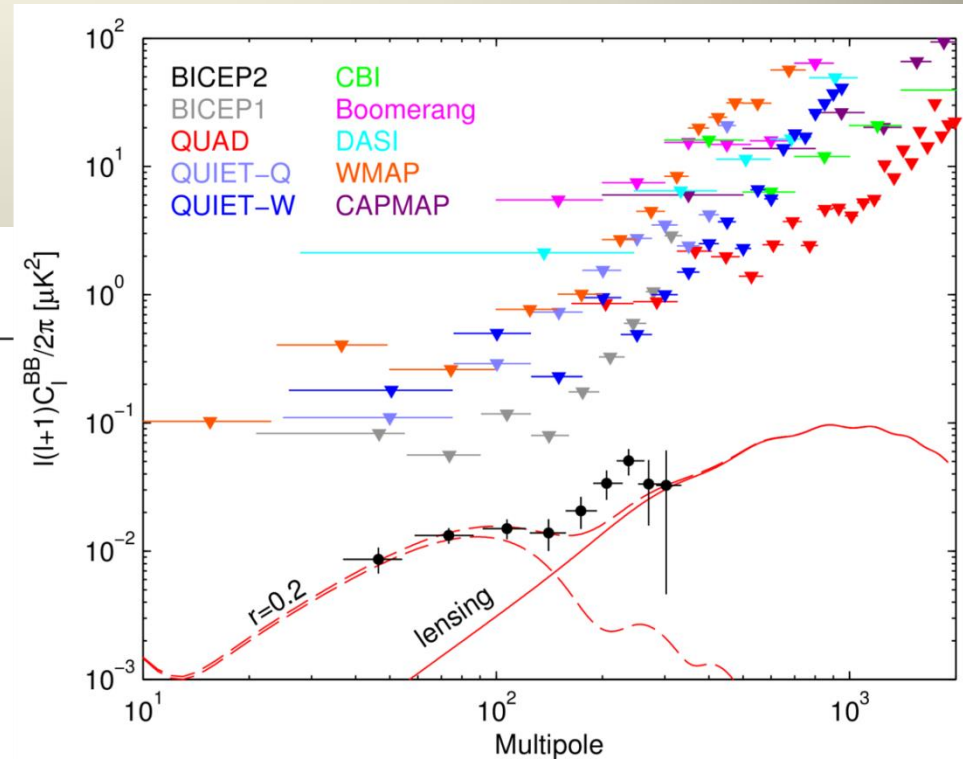
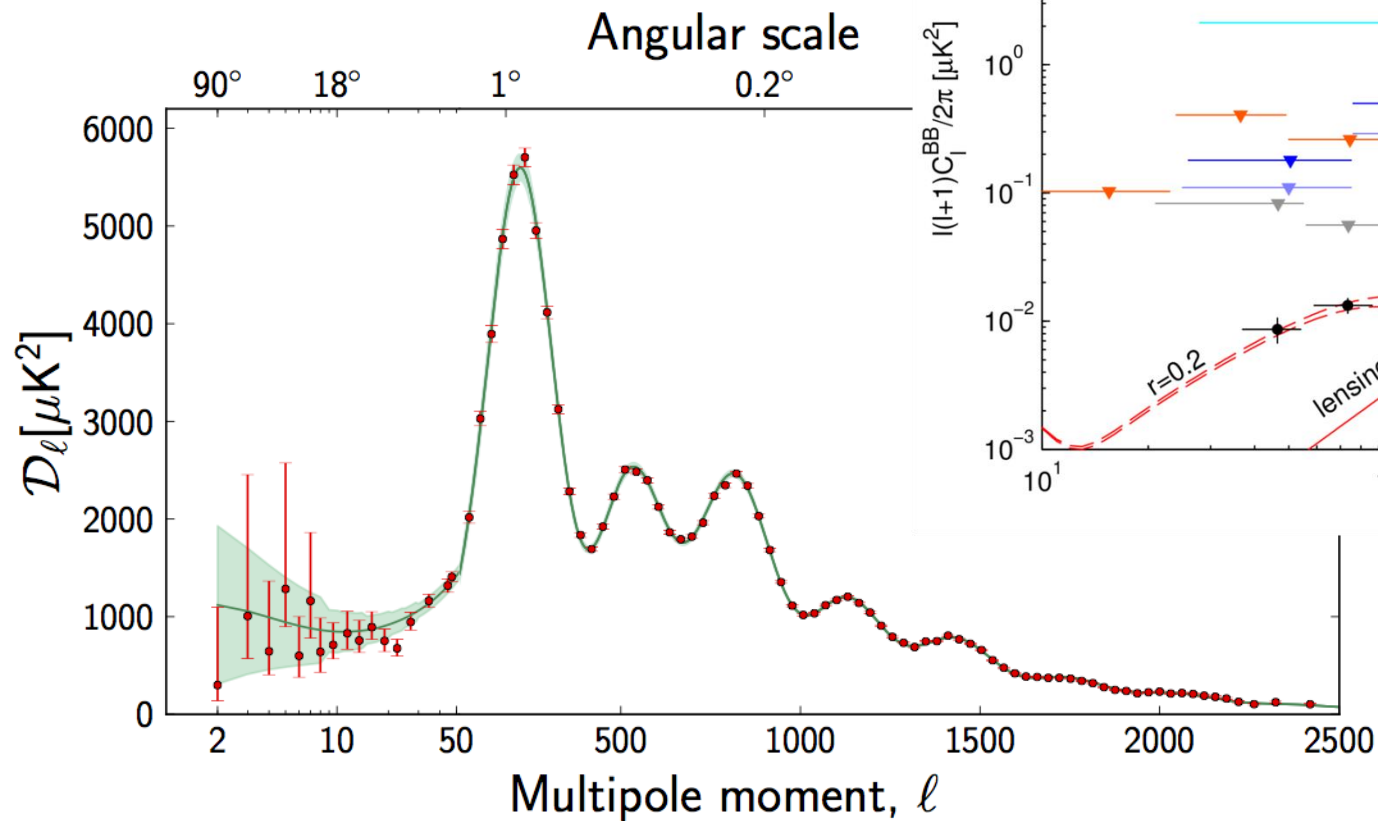


Polarisation and spatial curvature

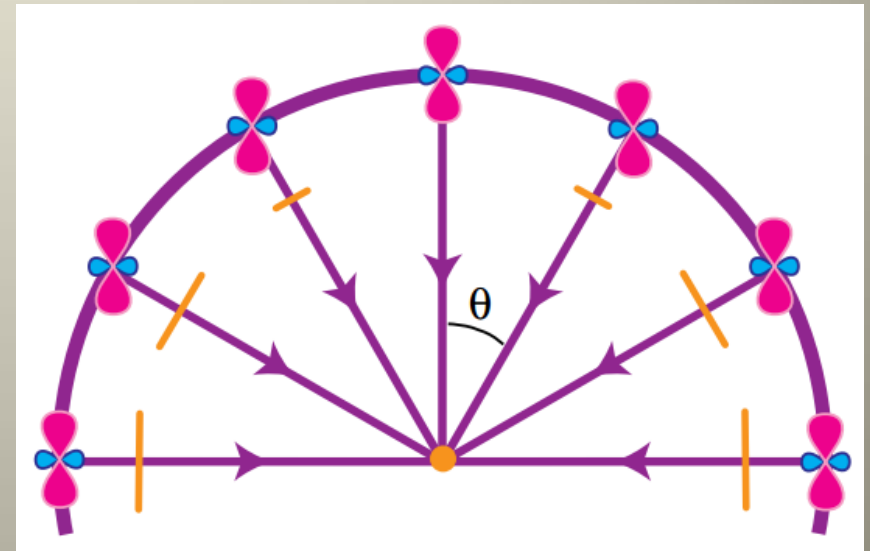
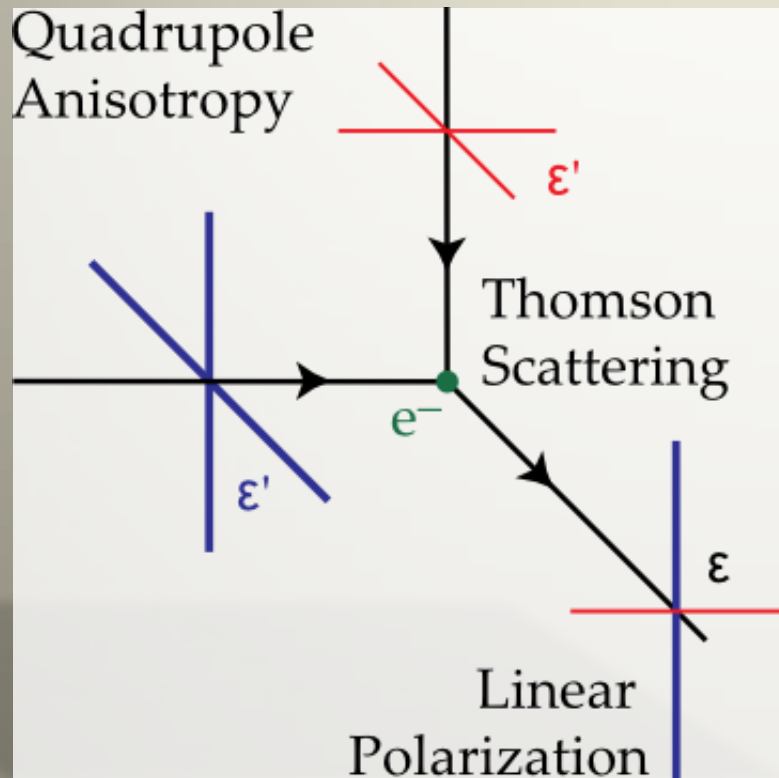
Thomas Tram

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CMB observables



CMB polarisation



Credits: Hu&White, astro-ph/9706147

Stokes parameters

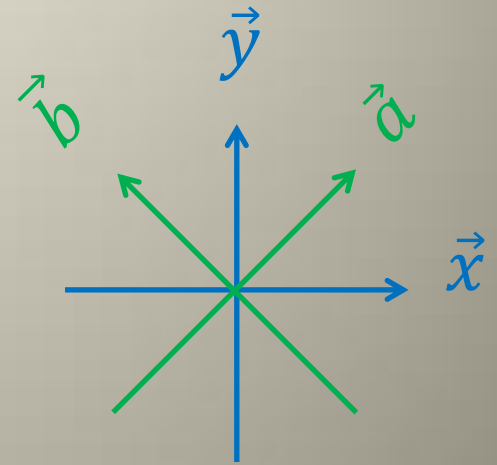
- A general radiation field is described by the 4 Stokes parameters:

$$I = \langle E_x^2 \rangle + \langle E_y^2 \rangle$$

$$Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle$$

$$U = \langle E_a^2 \rangle - \langle E_b^2 \rangle$$

$$V = -2\text{Im}(\langle E_x E_y^* \rangle)$$



- They form the intensity matrix

$$\mathcal{J} = \begin{bmatrix} I + Q & U - \mathfrak{i}V \\ U + \mathfrak{i}V & I - Q \end{bmatrix} = I\mathbf{1} + Q\sigma_3 + U\sigma_1 + V\sigma_2$$

Stokes parameters

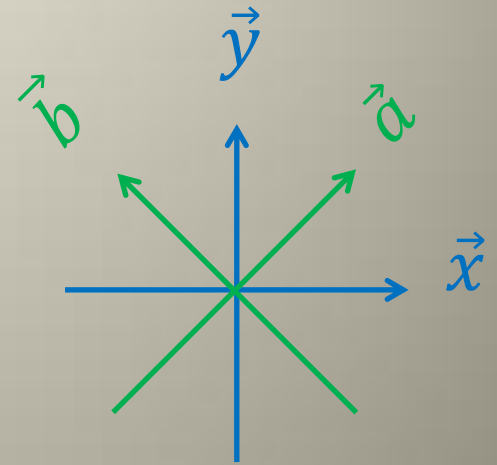
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- They form the intensity matrix

$$\mathcal{J} = \begin{bmatrix} I + Q & U \\ U & I - Q \end{bmatrix} = I\mathbf{1} + Q\sigma_3 + U\sigma_1$$

E and B polarisation $\mathcal{J} = \begin{bmatrix} I + Q & U \\ U & I - Q \end{bmatrix}$

- Rank 2 tensor field \mathcal{J} on the sphere is covariant, not Q and U . (Why is I ?)
- But $Q \pm \mathbb{i}U \rightarrow e^{\mp 2\mathbb{i}\psi} (Q \pm \mathbb{i}U)$
- Expand in spin ± 2 spherical harmonics $_{\pm 2}Y_l^m$:
 $(Q \pm \mathbb{i}U)(\vec{x}, \vec{n}) \sim$
 $\int d^3k e^{\mathbb{i}\vec{k} \cdot \vec{x}} \sum (-\mathbb{i})^l \left\{ E_l^{(m)} \pm \mathbb{i} B_l^{(m)} \right\} _{\pm 2}Y_l^m$

Photon Boltzmann equation

- Complicated because of geometry:
 - Simple derivation using ${}_sY_l^m$ harmonics thanks to Hu&White: astro-ph/9702170
 - Use vector $\vec{T} = (\Theta, Q + \mathring{U}, Q - \mathring{U})$
- Boltzmann equation:

$$\frac{d\vec{T}}{d\tau} = \frac{\partial \vec{T}}{\partial \tau} + n^i \vec{T}_{|i} = \vec{C}[\vec{T}] + \begin{pmatrix} D_\Theta \\ 0 \\ 0 \end{pmatrix}$$

Photon Boltzmann equation II

$$\frac{d}{d\tau} \begin{pmatrix} \Theta \\ Q + \frac{1}{2}U \\ Q - \frac{1}{2}U \end{pmatrix} + \dot{\kappa} \begin{pmatrix} \Theta - \hat{n} \cdot \vec{v}_b - \int \frac{d\Omega'}{4\pi} \Theta' \\ Q + \frac{1}{2}U \\ Q - \frac{1}{2}U \end{pmatrix} - \begin{pmatrix} D_\Theta \\ 0 \\ 0 \end{pmatrix} = \frac{\dot{\kappa}}{10} \int d\Omega'$$

$$\sum_{m=-2}^2 \begin{pmatrix} Y_2^{m'} Y_2^m & -\sqrt{\frac{3}{2}} Y_2^{m'} Y_2^m & -\sqrt{\frac{3}{2}} Y_2^{m'} Y_2^m \\ -\sqrt{6} Y_2^{m'} Y_2^m & 3 Y_2^{m'} Y_2^m & 3 Y_2^{m'} Y_2^m \\ -\sqrt{6} Y_2^{m'} Y_2^m & 3 Y_2^{m'} Y_2^m & 3 Y_2^{m'} Y_2^m \end{pmatrix} \begin{pmatrix} \Theta' \\ Q' + \frac{1}{2}U' \\ Q' - \frac{1}{2}U' \end{pmatrix}$$

- Details not important, but the structure is.

Hu&White: astro-ph/9702170

Hu,White,Seljak,Zaldarriaga: astro-ph/9709066

Photon Boltzmann equation III

$$\frac{d}{d\tau} \begin{pmatrix} \Theta \\ Q \\ \ddot{U} \end{pmatrix} + \dot{\kappa} \begin{pmatrix} \Theta - \hat{n} \cdot \vec{v}_b - \int \frac{d\Omega'}{4\pi} \Theta' \\ Q \\ \ddot{U} \end{pmatrix} - \begin{pmatrix} D_\Theta \\ 0 \\ 0 \end{pmatrix} =$$

$$\frac{\dot{\kappa}}{10} \sum_{m=-2}^2 \int d\Omega' \begin{pmatrix} Y_2^m \left\{ Y_2^{m'} \Theta' - \sqrt{\frac{3}{2}} \mathcal{E}^{m'} Q' - \sqrt{\frac{3}{2}} \mathcal{B}^{m'} \ddot{U}' \right\} \\ \frac{1}{2} \mathcal{E}^m \left\{ -\sqrt{6} Y_2^{m'} \Theta' + 3 \mathcal{E}^{m'} Q' + 3 \mathcal{B}^{m'} \ddot{U}' \right\} \\ \frac{1}{2} \mathcal{B}^m \left\{ -\sqrt{6} Y_2^{m'} \Theta' + 3 \mathcal{E}^{m'} Q' + 3 \mathcal{B}^{m'} \ddot{U}' \right\} \end{pmatrix}$$

where $\mathcal{E}^m \stackrel{\text{def}}{=} {}_2 Y_2^m + {}_{-2} Y_2^m$ and $\mathcal{B}^m \stackrel{\text{def}}{=} {}_2 Y_2^m - {}_{-2} Y_2^m$.

But since this equation holds separately for each m...

Photon Boltzmann equation IV

...we must have $\dot{\mathbb{U}}^{(m)} = \frac{\mathcal{B}^m}{\varepsilon^m} Q^{(m)}!$

$$\frac{d}{d\tau} \begin{pmatrix} \Theta^{(m)} \\ Q^{(m)} \end{pmatrix} + \dot{\kappa} \begin{pmatrix} \Theta^{(m)} - \hat{n} \cdot \vec{v}_b^{(m)} - \int \frac{d\Omega'}{4\pi} \Theta' \\ Q^{(m)} \end{pmatrix} - \begin{pmatrix} D_{\Theta} \\ 0 \end{pmatrix} =$$

$$\frac{\dot{\kappa}}{10} \int d\Omega' \begin{pmatrix} Y_2^m \left\{ Y_2^{m'} \Theta' - \sqrt{\frac{3}{2}} \left[\varepsilon'^m + \frac{(\mathcal{B}^{m'})^2}{\varepsilon'^m} \right] Q' \right\} \\ - \sqrt{\frac{3}{2}} \varepsilon'^m \left\{ Y_2^{m'} \Theta' - \sqrt{\frac{3}{2}} \left[\varepsilon'^m + \frac{(\mathcal{B}^{m'})^2}{\varepsilon'^m} \right] Q' \right\} \end{pmatrix}$$

J.Lesgourgues&TT:arXiv:1305.3261

Relation to E and B

- Line-of-sight solutions needed for efficient implementation
 - CMBfast, Seljak&Zaldarriaga 1996
 - Known for scalars, vectors and tensors in non-flat universes in terms of Θ_l , E_l and B_l (HWSZ 1998)
 - Done if we can relate the multipole moments of $\Theta_l^{(m)}$, $E_l^{(m)}$ and $B_l^{(m)}$ to $F_l^{(m)}$ and $G_l^{(m)}$

6 months later...

It can be done!

$$\Theta_l^{(0)} = \frac{2l+1}{4} F_l^{(0)}$$

$$E_l^{(0)} = \frac{2l+1}{4} F_l^{(0)} \sqrt{\frac{(l-2)!}{(l+2)!}} \left(-l(l-1) G_l^{(0)} + \sum_{\substack{k=0, \\ k+l \text{ even}}}^{l-2} 2^{\frac{l-k}{2}} (2k+1) G_k^{(0)} \right)$$

∴ (For vector modes see paper)

$$\Theta_l^{(2)} = -\frac{1}{4} \sqrt{\frac{(l+2)!}{(l-2)!}} \left(\frac{1}{2l-1} F_{l-2}^{(2)} + \frac{2(2l+1)}{(2l-1)(2l+3)} F_l^{(2)} + \frac{1}{2l+3} F_{l+2}^{(2)} \right)$$

$$E_l^{(2)} = \sqrt{\frac{2l+1}{5}} \left(-(2l-3) \alpha_{l-2}^l G_{l-2}^{(2)} + (2l+1) \alpha_l^l G_l^{(2)} - (2l+5) \alpha_{l+2}^l G_{l+2}^{(2)} \right)$$

$$B_l^{(2)} = \sqrt{\frac{2l+1}{5}} \left((2l-1) \alpha_{l-1}^l G_{l-1}^{(2)} - (2l+3) \alpha_{l+1}^l G_{l+1}^{(2)} \right)$$

The line-of-sight integrals

- We have computed the source, now convolve with certain radial functions:

$$\frac{\Theta_l^{(m)}}{2l+1} = \sum_j \int_0^{\tau_0} d\tau e^{-\kappa} \mathcal{S}_j^{(m)} \phi_l^{(jm)}$$
$$\frac{E_l^{(m)}}{2l+1} = - \int_0^{\tau_0} d\tau \dot{\kappa} e^{-\kappa} \sqrt{6} P^{(m)} \epsilon_l^{(m)}$$
$$\frac{B_l^{(m)}}{2l+1} = - \int_0^{\tau_0} d\tau \dot{\kappa} e^{-\kappa} \sqrt{6} P^{(m)} \beta_l^{(m)}$$

Spatial curvature

- The radial functions are linear combinations of Φ_l^ν , $\Phi_l^{\nu'}$ and $\Phi_l^{\nu''}$.
- In the flat limit: j_l , ν dependence becomes a rescaling of argument. ($x = k(\tau_0 - \tau)$)
- During MCMC, we must compute them on the fly => Much longer execution time
- However: Flat rescaling approximation!

Hypergeometric Bessel functions II

- FLRW metric:

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right],$$

$$r(\chi) = \begin{cases} \sin \chi & K = 1 \\ \chi & K = 0 \\ \sinh \chi & K = -1 \end{cases}$$

- Radial part of $\nabla^2 f = -k^2 f$ equivalent to:

$$u'' = \left[\frac{l(l+1)}{r(\chi)^2} - v^2 \right] u \text{ where } u = r(\chi)\Phi(\chi)$$

What is going on???

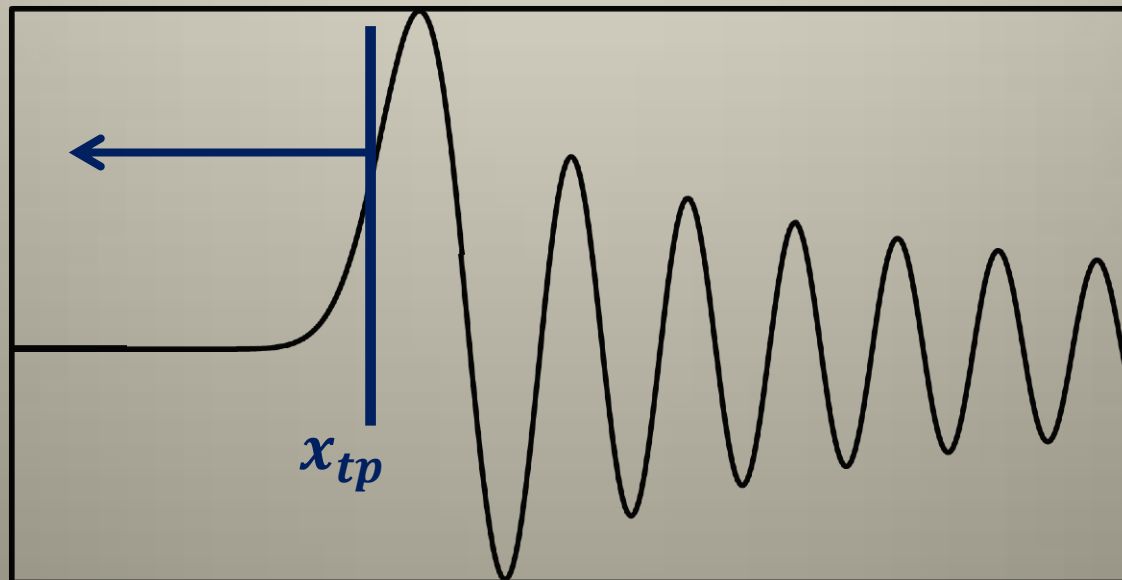
- The Boltzmann equation is a Partial Differential Equation (PDE):

$$\frac{d\vec{T}}{d\tau} = \frac{\partial \vec{T}}{\partial \tau} + n^i \vec{T}_{|i} = \vec{C}[\vec{T}] + \begin{pmatrix} D_{\Theta} \\ 0 \\ 0 \end{pmatrix}$$

- We do not like to solve PDEs, only ODEs. So we employ a *spectral method*: we expand \vec{T} in eigenfunctions of the differential operator.

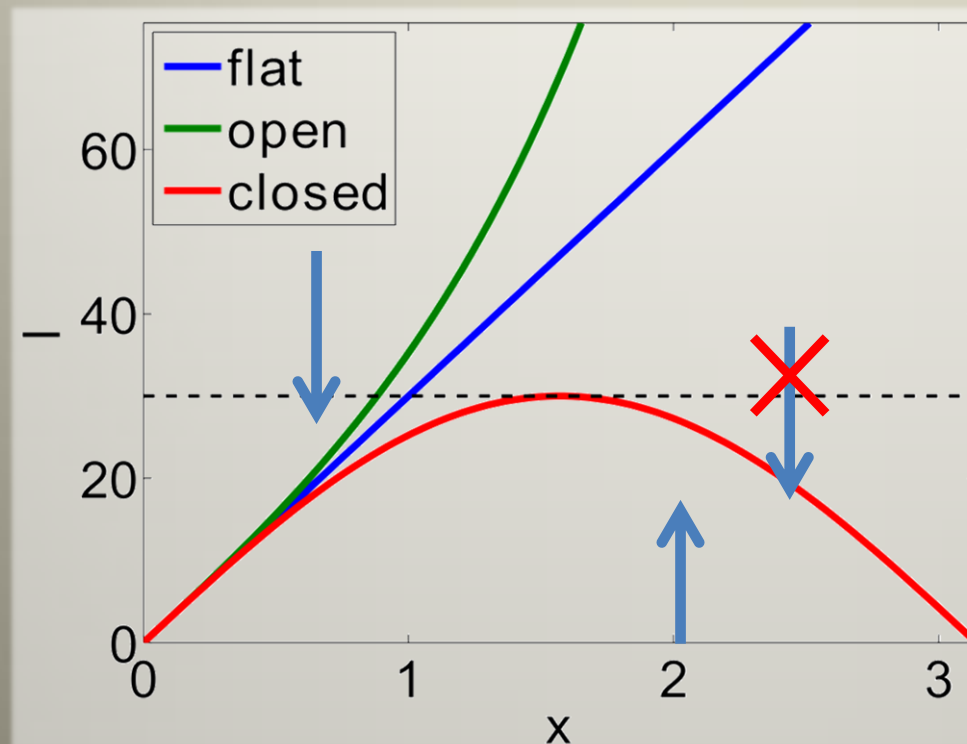
Schrödinger-like equation

- We have a classical turningpoint, separating the dissipative region $0 < x < x_{tp}$ from the dispersive region $x_{tp} < x < \infty$



Standard recurrence method

- Backwards recurrence in dissipative region, forwards recurrence in dispersive region.



But:
Closed model
restricted by
 $l < \nu$, so no
backwards
recurrence...

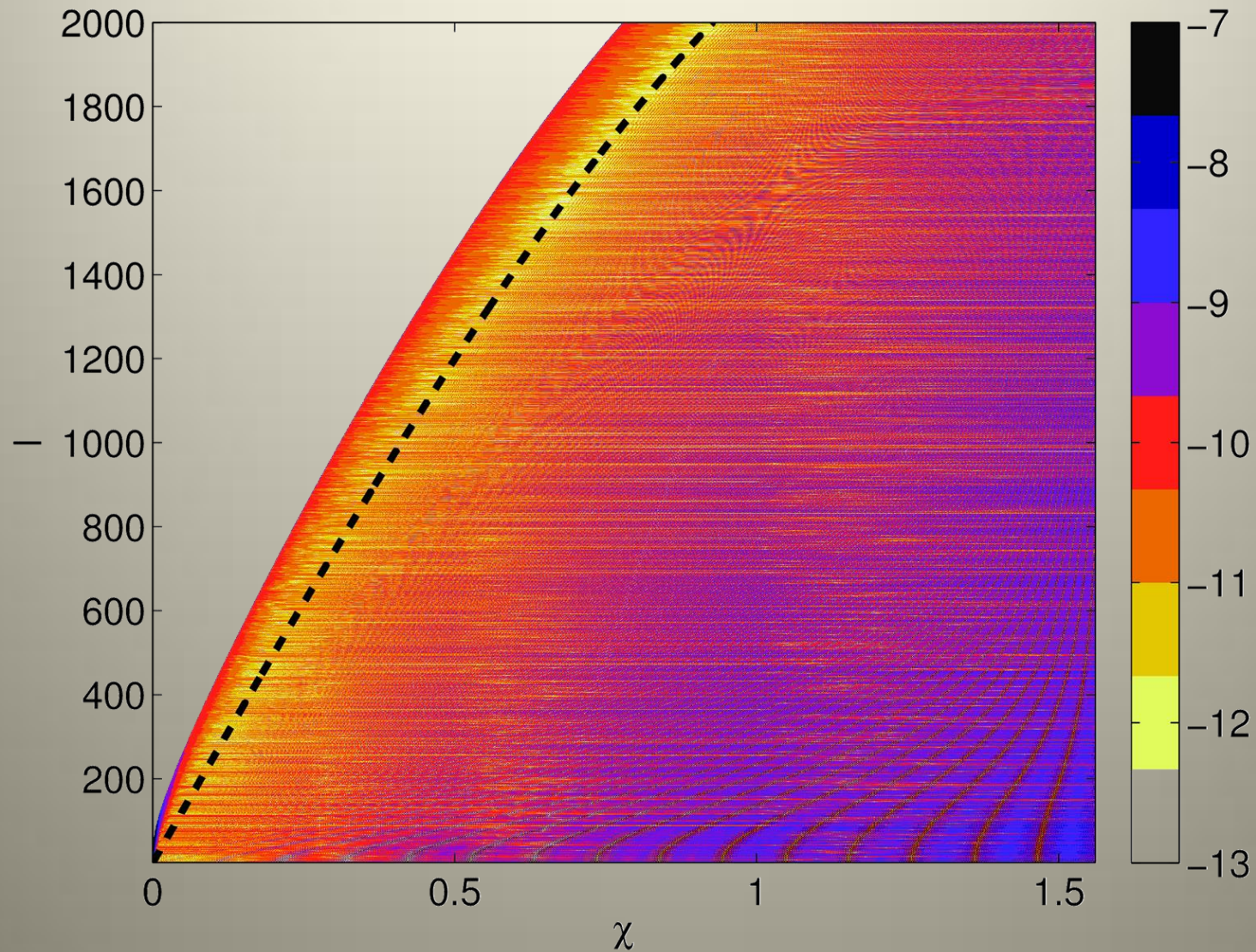
Relation to Gegenbauer polynomials

- We found the following important identity:

$$\Phi_l^\nu(x) = 2^l l! \sqrt{\frac{(\nu-l-1)!}{\nu(\nu+l)!}} \sin^l(x) C_{\nu-l-1}^{l+1}(\cos(x))$$

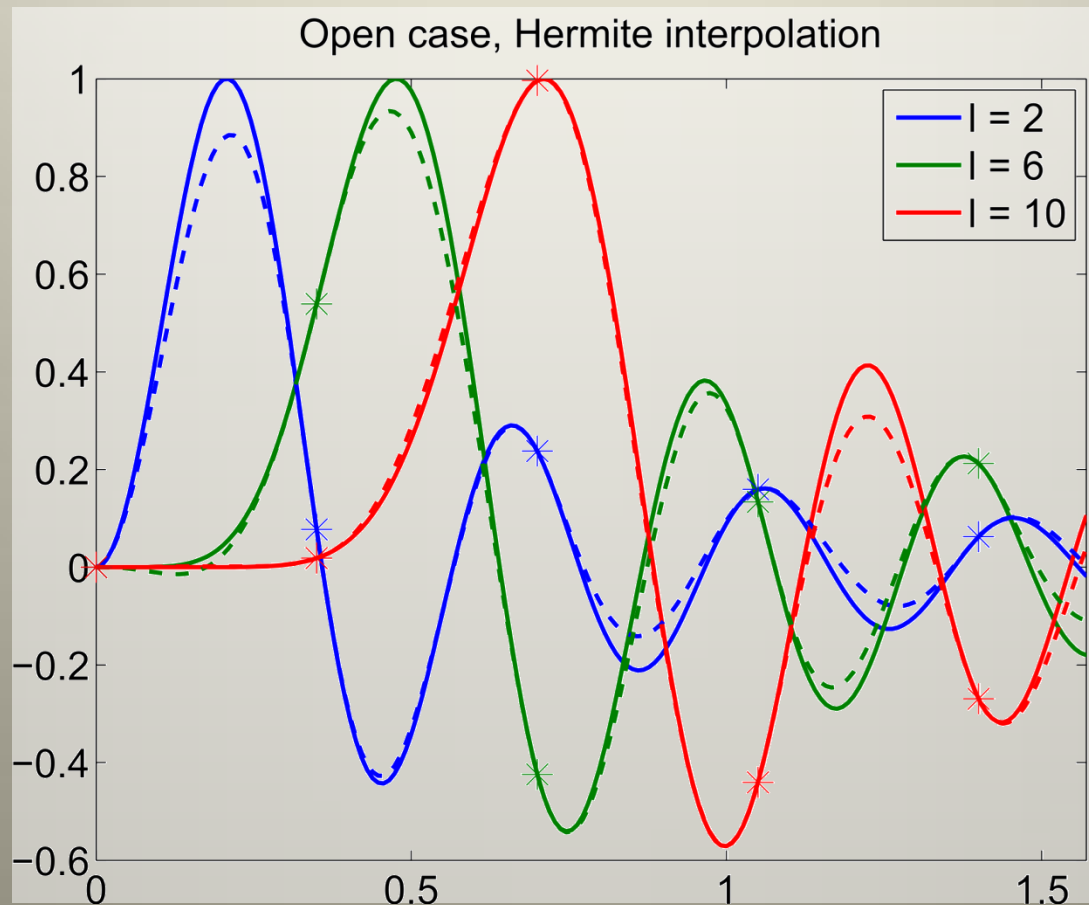
- Consider $l = \nu - 1$, then $C_0^{l+1}(\cos(x)) = 1$,
and we have downwards recurrence!

It works!



Hermite interpolation

- Store Φ and Φ' , compute higher order derivatives.
- Use 6 constraints \Rightarrow 5th order polynomial



**Note: Only 5
computed
points!**