# Cosmological Parameter Extraction from Data

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# Outline

1. Introduction
   - Need for a statistical tool
   - Bayesian approach
2. Comparison of existing methods
   - Metropolis-Hastings
   - EEMCE
   - Nested Sampling

# Outline

# Bibliography

## Book

- Bayesian methods in cosmology (Hobson, Jaffe, Liddle, Mukherjee and Parkinson)

## Review Articles

- Comparison of sampling techniques for Bayesian parameter estimation (*Rupert Allison, Joanna Dunkley*, arXiv:1308.2675)
- Bayes in the sky: Bayesian inference and model selection in cosmology (*Roberto Trotta*, arXiv:0803.4089)

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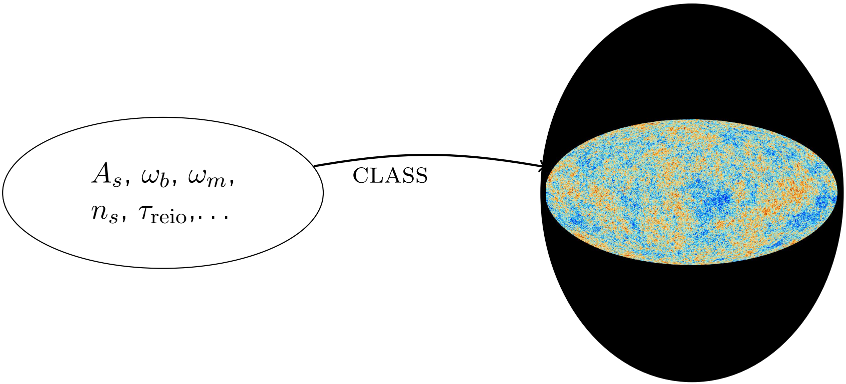
## Similar to Particle Physics

Prediction only for, *e.g.* the rate of decay of a particle. Information acquired when statistically observing this decay channel (how many times did it decay to this particular product ?)

# The big picture



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## from Bayesian Methods in Cosmology

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- The question we can answer is: **is this model better than another ?**
- We infer **credible regions** in which, given a model, the parameters live.
- Our knowledge depends on the **data measured**.

# Bayesian approach

## Definitions

Given a **context** $\mathcal{I}$ and a set of **data** $D$:

$$\left.\begin{array}{l}\bullet\ \theta\text{: continuous values for a parameter set} \\ \bullet\ \mathrm{pr}(\theta) \geq 0\text{: probability function is positive} \\ \bullet\ \int \mathrm{pr}(\theta)\mathrm{d}\theta = 1\text{: Sum rule} \\ \bullet\ \mathrm{pr}(\phi, \theta) = \mathrm{pr}(\phi|\theta)\mathrm{pr}(\theta)\text{: Product rule}\end{array}\right\} \ || \ \mathcal{I}$$

# Bayes Theorem

## Bayes Theorem

*All quantities are given in the context $\mathcal{I}$*

$$\begin{array}{ccccc}
\mathrm{pr}(\theta)\mathrm{pr}(D|\theta) & = & \mathrm{pr}(\theta, D) & = & \mathrm{pr}(D)\mathrm{pr}(\theta|D) \\
\textbf{Prior} \times \textbf{Likelihood} & = & \textbf{Joint} & = & \textbf{Evidence} \times \textbf{Posterior} \\
\pi(\theta)\mathcal{L}(\theta) & = & \dots & = & E\mathcal{P}(\theta) \\
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## Prior

*A priori* information on the parameters. Most of the time, it is **flat** (uniform chance to be inside a given volume).

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## Likelihood

Given by the instrument operated at *known input*. If uncontrolled unknow: **nuisance parameters**.

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## Evidence

Only recovered with Nested Sampling, gives an information on how well the given context suits the data.

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## Posterior Distribution

The name of the game. Inferred distribution of probability, after using the data.

# Bayes Theorem

## Complete Rules

$$\int \pi(\theta)\mathrm{d}\theta = \int \mathcal{P}(\theta)\mathrm{d}\theta = 1$$

$$E = \int \mathcal{L}(\theta)\pi(\theta)\mathrm{d}\theta$$

$$\mathcal{P}(\theta) = \frac{\pi(\theta)\mathcal{L}(\theta)}{E}$$

# Bayesian approach: issues

## What are the problems ?

- **Evidence** is hard to compute because. . .
- we don't usually know the **Likelihood** function (analytically).
- so we don't know where to **sample** it (many dimensions). . .
- and it might be **computationally expensive**.

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- and it might be **computationally expensive**.

## How to deal with it ?

- we have to sample **randomly** the volume (*all methods*)
- we can avoid computing this integral by **computing ratios** (*mcmc*)

# Methods that give the Evidence

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Evidence is **how well your context explains the data**. Nested Sampling gives this. Others dont... but best-fit likelihood gives some indication.

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## Caution

These methods will only give you a **probable** answer, not a definite one. It is the price you pay for not doing your integral exactly.

# But. . .

## Living without the evidence

**Theoretically motivated model**: we want to know the values of this parameter to explain the data. Then, the **posterior** is an interesting quantity, and the **evidence** can be left aside temporarily.

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## Nonetheless

Beware of the best-fit likelihood value !

# Outline

# Algorithm and Sampler

## Foreword

### Algorithm

Tells you how to **choose** which point you move, and if you **accept** a new point or not

### Sampler

Tells you how to **move**, **select a** new point.

Sometimes used **interchangeably**

# Algorithm and Sampler

## Foreword

**available in Monte Python**

As of v2.0.0, you can use **MultiNest** (`nested sampling`, by Farhan Feroz & Mike Hobson), and the **CosmoHammer** (`emcee`, by Joel Akeret & Sebastian Seehars)

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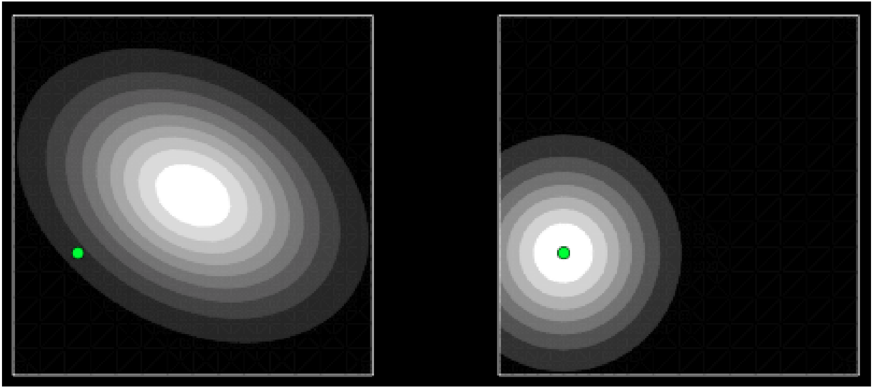
### Cosmo Hammer

Thanks to Joel and Sebastian for helping setting this up

# Metropolis-Hastings

## Courtesy of Sebastian Seehars



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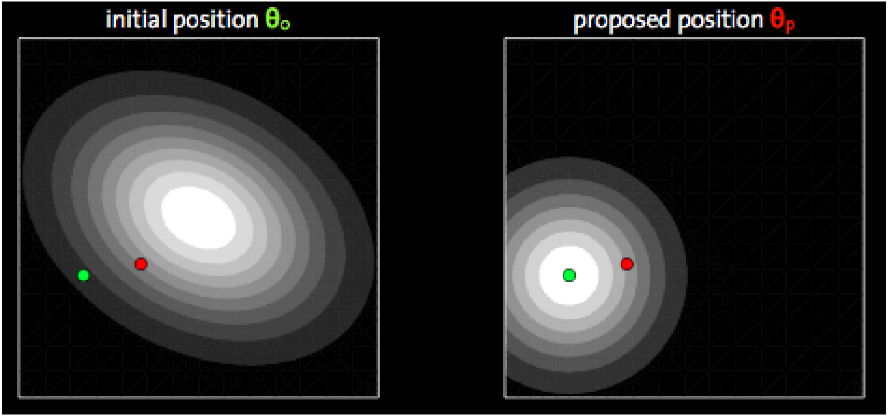

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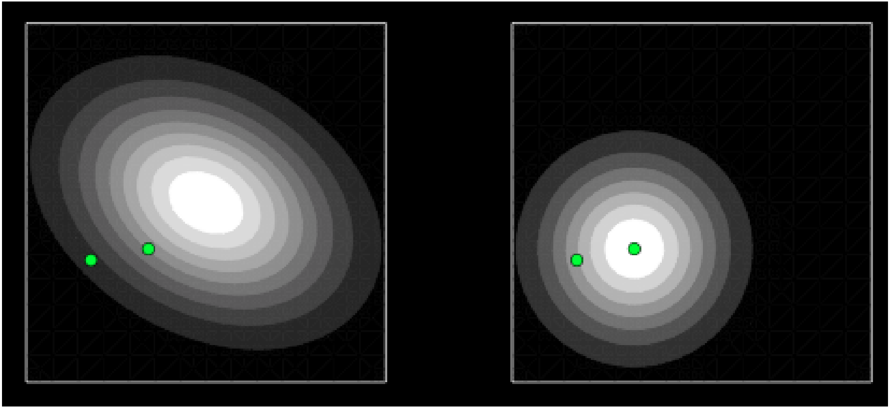

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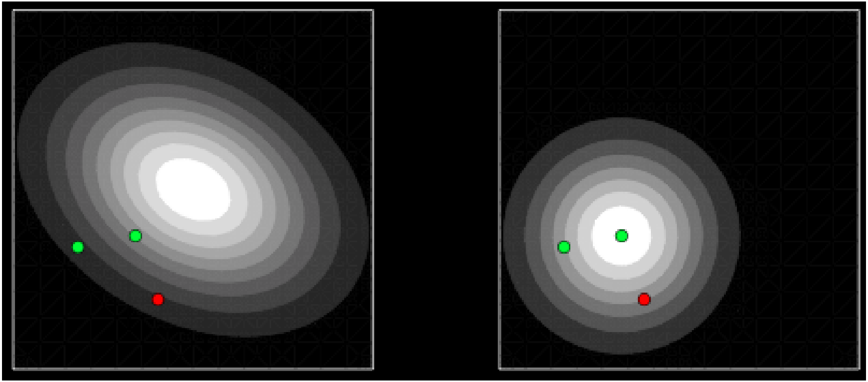

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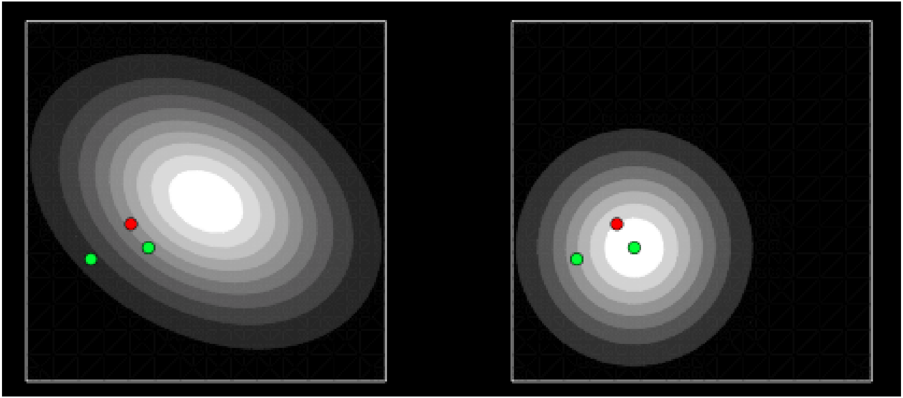

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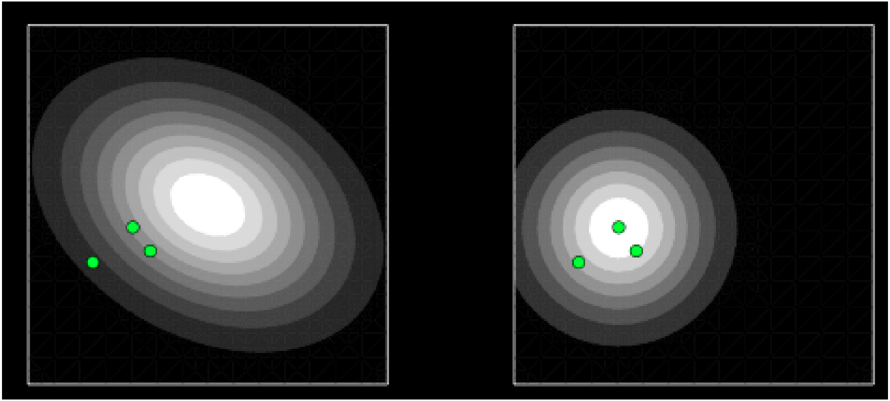

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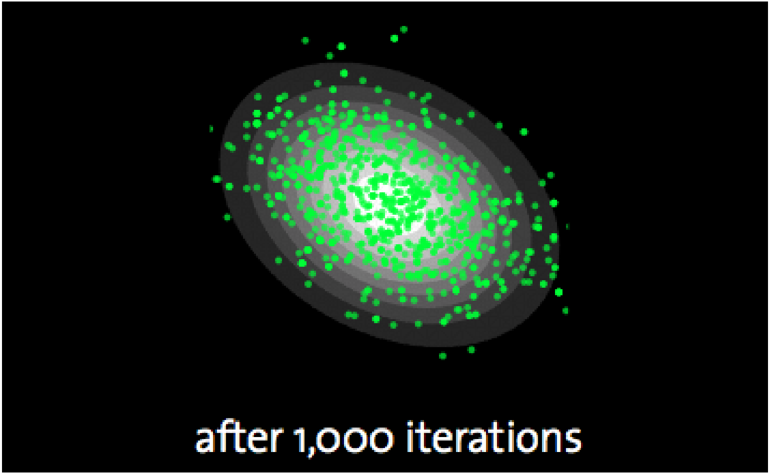

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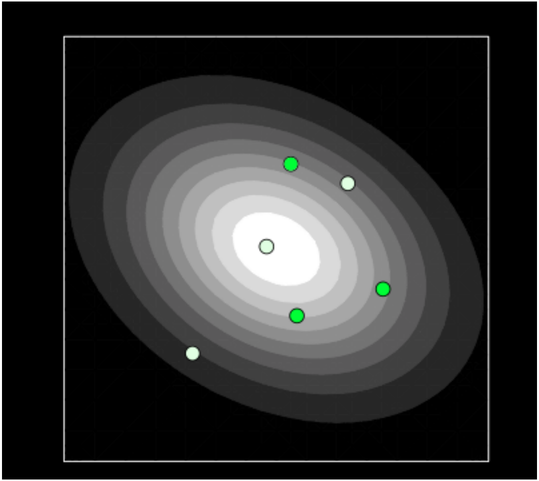

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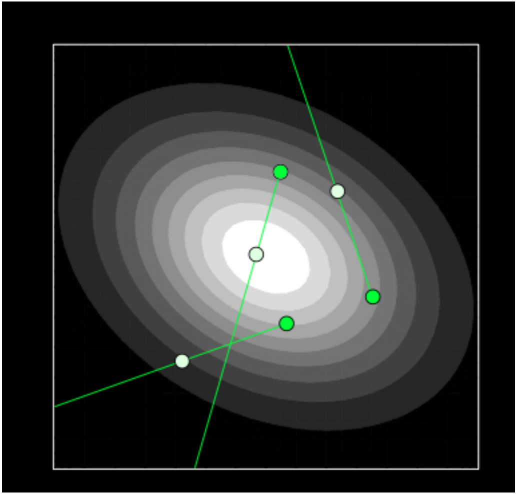

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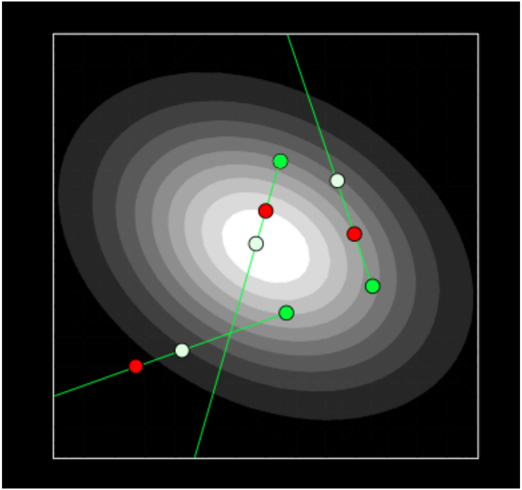

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## Pros/Cons

**Good to explore weirdly shaped distributions**,
**but scales not so well with many parameters**

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- Iterate over the points: **discard least likely**, pick a new point until a **more likely** is found
- To propose a new point: **approximate the posterior distribution by encompassing all other points within an ellipse**.
- The sorting of these points by **increasing likelihood** allows to compute the evidence ($\simeq$ trapezoidal integration)

# Nested Sampling

## Mathematical Procedure

$$E = \int \mathcal{L}(\theta)\pi(\theta)\mathrm{d}\theta$$

$$dX = \pi(\theta)\mathrm{d}\theta\text{: prior volume}$$

$$X(\lambda) = \int_{\mathcal{L}(\theta)>\lambda} \pi(\theta)\mathrm{d}\theta$$

$$E = \int_0^1 \mathcal{L}(X)dX$$

$\mathcal{L}(X)$ is a monotonically decreasing function of $X$

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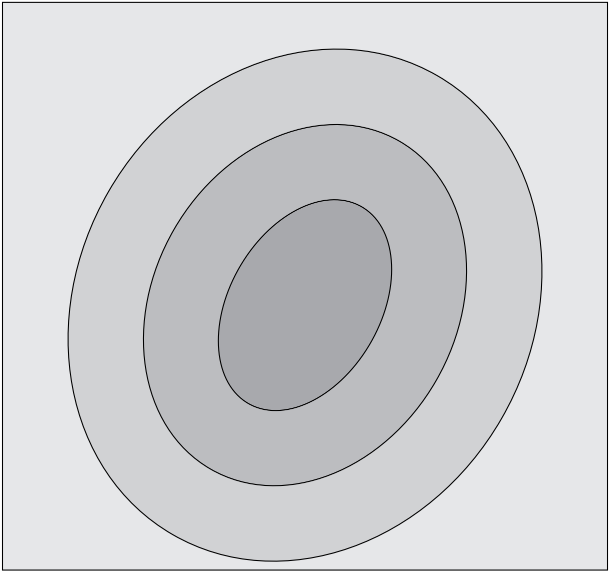

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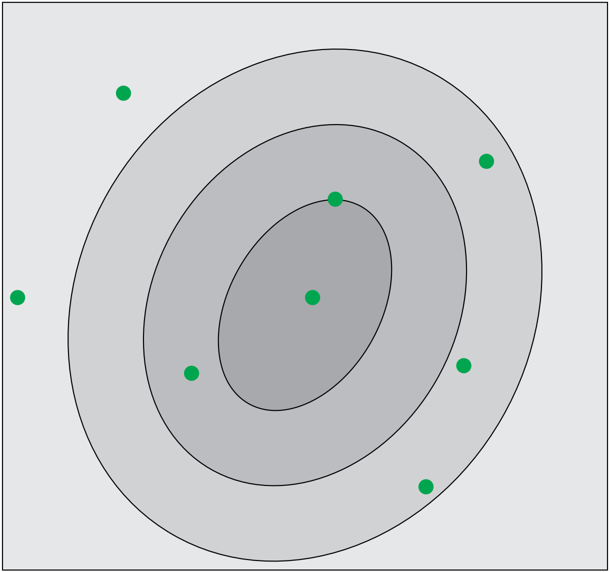

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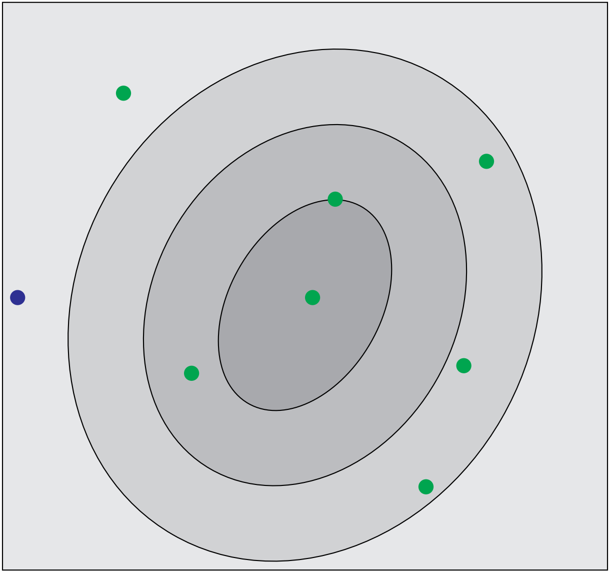

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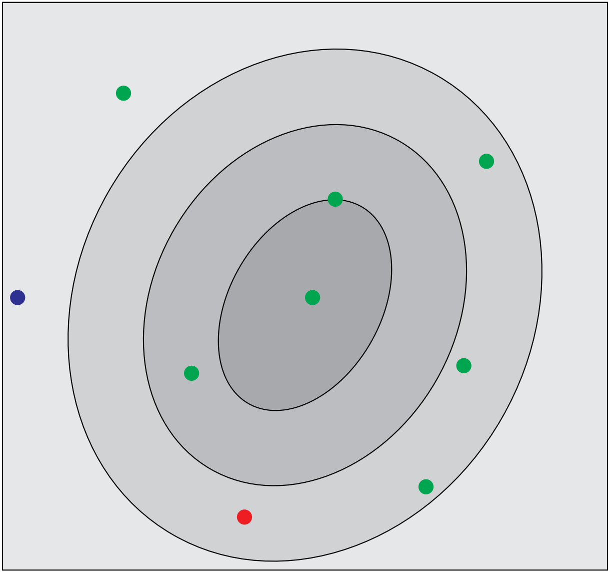

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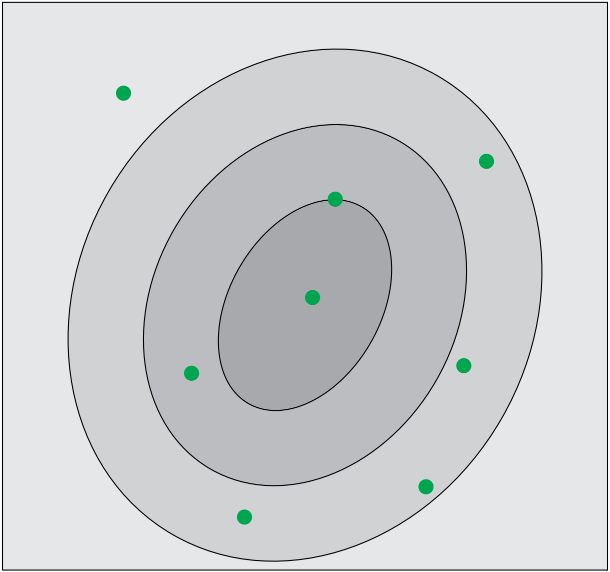

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- The integral $E$ is computed with the **active set**
- The **prior volume** shrinks on the most likely points
- Stops when $\Delta E < 0.5$